# Advanced Particle Physics

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Cours de l'École Polytechnique
Master 2 - PHE

September 2023

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# Fermions

## Fermions

Matter = fermions
(Spin-$\frac{1}{2}$ particles):

- Electrons with two spin orientations: L and R
- Neutrinos (L)
- Quarks L and R (proton=uud, neutron=udd)
- Three families = heavier copies of the first family

## Particle

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# ... and Bosons

## Bosons

Interactions = bosons (Spin–0 or –1 particles):

- Electromagnetism: Spin–1 massless
- Strong interaction (p=uud): Spin–1 massless
- Weak interaction: Spin–1 massive
- Masses: Spin–0 massive

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# Electric charge

- Fractional charges not observed in nature
- Strong interaction: uud, udd

## Properties

$$\begin{array}{r} \frac{2}{3} \\ -\frac{1}{3} \end{array} \quad \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \begin{pmatrix} c_L \\ s_L \end{pmatrix} \quad \begin{pmatrix} t_L \\ b_L \end{pmatrix}$$

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# Colour charge

## Charges

- Sum of colours (RGB) white
- R+G+B= (qqq =baryon)
- Colour+anti-colour= White ($q\bar{q}$ =meson)
- Gluon carries colour+anti-colour
- 8 different gluons (not 9)

## Properties

$$\begin{array}{cccc}
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C + \bar{C}' & g & & \\
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- & H & &
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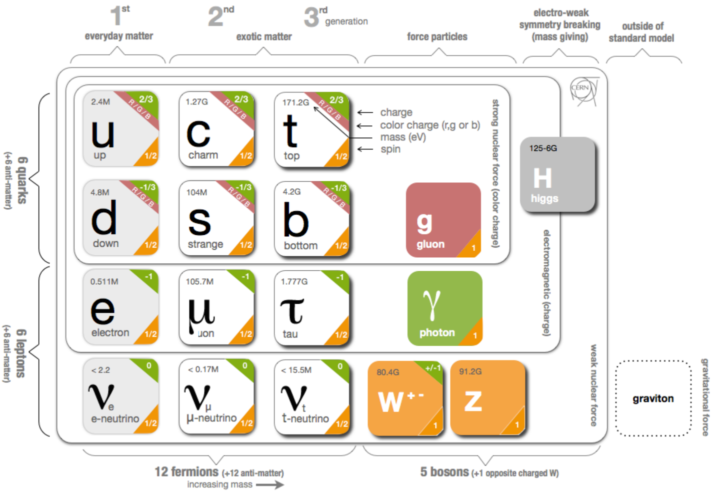
# Standard Model Overview



# Interaction intensities

## Rule of thumb for interactions

| Interaction | Carrier | Relative strength |
|---|---|---|
| Gravitation | Graviton (G) | $10^{-40}$ |
| Weak | Weak Bosons ($W^{\pm}$,$Z^{\circ}$) | $10^{-7}$ |
| Electromagnetic | Photon ($\gamma$) | $10^{-2}$ |
| Strong | Gluon (g) | 1 |

- Forget about Gravitation in particle physics problems
- The course will lead us to understand how the model describes the interactions and their strength.

# Metric and Lorentz transformation

## Metric

A four-vector **x** is attributed to a particular space-time point.

$$\mathbf{x} = (x^\mu) = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$$

Greek letters are for four-vectors
Roman letters for spatial coordinates (vectors)
The scalar product is defined thanks to the metric tensor $g^{\mu\nu}$

$$\mathbf{g} = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

by

$$\mathbf{x}.\mathbf{y} = g_{\mu\nu} x^\mu y^\nu = x^\mu y_\mu = x_\mu y^\mu$$

# Metric and Lorentz transformation

## Lorentz transformation

- A transformation $(\Lambda, a)$ defines the transition from an inertia frame to another

$$(\Lambda, a) : x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}$$

- The energy and 3-momentum $p$ of a particle of mass m form a four-vector whose square $\mathbf{p}.\mathbf{p} = m^2$
- In the course, we will apply the Einstein summation rule on greek indices
- The velocity of the particle is $\beta = v/c = p/E$
- and the Lorentz factor is $\gamma = \frac{1}{\sqrt{1-\beta^2}}$
- The energy and momentum $(E^{\star}, p^{\star})$ viewed from a frame moving with velocity $\beta_f$ are given by

$$\begin{pmatrix} E^{\star} \\ p^{\star}_{||} \end{pmatrix} = \begin{pmatrix} \gamma_f & -\gamma_f\beta_f \\ -\gamma_f\beta_f & \gamma_f \end{pmatrix} \begin{pmatrix} E \\ p_{||} \end{pmatrix}$$

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# Metric and Lorentz transformation

## Special relativity - Space-time coordinates

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

with

$$\beta = v/c \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

# Muon decay in the atmosphere

## Muon decay

Elementary particles called muons (which are identical to electrons, except that they are about 200 times as massive) are created in the upper atmosphere when cosmic rays collide with air molecules. The muons have an average lifetime of about $2 \times 10^{-6}$ seconds (then they decay into electrons and neutrinos), and move at nearly the speed of light.

Assume for simplicity that a certain muon is created at a height of 50 km, moves straight downward, has a speed $v = 0.99998c$, decays in exactly $T = 2 \times 10^{-6}$ seconds, and doesn't collide with anything on the way down. Will the muon reach the earth before it (the muon) decays ?

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## Solution

The naive thing to say is that the distance traveled by the muon is $d = vT \sim (3 \times 10^8 m/s)(2 \times 10^{-6} s)$ = 600m, and that this is less than 50 km, so the muon doesn't reach the earth.

This reasoning is incorrect, because of the time-dilation effect. The muon lives longer in the earth frame, by a factor $\gamma$, which is $\gamma = 1/\sqrt{(1 - v^2/c^2)} \sim 160$ here. The correct distance traveled in the earth frame is therefore $v\gamma T \sim 100km$. Hence, the muon travels the 50 km, with room to spare.

The real-life fact that we actually do detect muons reaching the surface of the earth in the predicted abundances (while the naived $vT$ reasoning would predict that we shouldn't see any) is one of the many experimental tests that support relativity.

# Muon decay in the atmosphere

## Muon decay again

Consider the “Muon decay” example. From the muon’s point of view, it lives for a time of $T = 2 \times 10^{-6}$ seconds, and the earth is speeding toward it at $v = 0.99998c$. How, then, does the earth (which travels only $d = vT \sim 600$ m before the muon decays) reach the muon?

# Muon decay in the atmosphere

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## Solution

The important point here is that in the muon’s frame, the distance to the earth is contracted by a factor $\gamma \sim 160$. Therefore, the earth starts only $50km/160 \sim 300m$ away. Since the earth can travel a distance of 600 m during the muon’s lifetime, the earth collides with the muon, with time to spare.

Time dilation and length contraction are intimately related. We can’t have one without the other.

# The Mandelstam variables

$$\mathbf{a} = (E_a, \vec{p}_a) = (p_0, p_1, p_2, p_3)$$
$$E_a \cdot E_a - \vec{p}_a \cdot \vec{p}_a = m_a^2$$
$$g^{\mu\nu} p_\mu p_\nu = m_a^2$$

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## Conservation of E and $\vec{p}$

$$\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$$

therefore

$$\mathbf{a} - \mathbf{c} = \mathbf{d} - \mathbf{b}$$

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## Mandelstam Variables

$$a + b \to c + d$$

$$s = (\mathbf{a} + \mathbf{b})^2$$

$$t = (\mathbf{a} - \mathbf{c})^2$$

$$u = (\mathbf{a} - \mathbf{d})^2$$

# The Mandelstam variables

## Theorem

$$s + t + u = \quad m_a^2 + m_b^2 + m_c^2 + m_d^2$$

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**Proof.**

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## Proof.

$$s + t + u = \mathbf{a}^2 + \mathbf{b}^2 + 2 \cdot \mathbf{a} \cdot \mathbf{b} + \mathbf{a}^2 + \mathbf{c}^2 - 2 \cdot \mathbf{a} \cdot \mathbf{c} + \mathbf{a}^2 + \mathbf{d}^2 - 2 \cdot \mathbf{a} \cdot \mathbf{d}$$

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$$\begin{aligned} s + t + u &= \mathbf{a}^2 + \mathbf{b}^2 + 2 \cdot \mathbf{a} \cdot \mathbf{b} + \mathbf{a}^2 + \mathbf{c}^2 - 2 \cdot \mathbf{a} \cdot \mathbf{c} + \mathbf{a}^2 + \mathbf{d}^2 - 2 \cdot \mathbf{a} \cdot \mathbf{d} \\ &= 3m_a^2 + m_b^2 + m_c^2 + m_d^2 + 2 \cdot \mathbf{a}\,(\mathbf{b} - \mathbf{c} - \mathbf{d}) \\ &= 3m_a^2 + m_b^2 + m_c^2 + m_d^2 - 2 \cdot \mathbf{a}\,(\mathbf{a}) \\ &= 3m_a^2 + m_b^2 + m_c^2 + m_d^2 - 2m_a^2 \\ &= m_a^2 + m_b^2 + m_c^2 + m_d^2 \end{aligned}$$

□

2 particle reaction → 2 independent variables!

# The Mandelstam variables

## Useful relationships

$$t = -2(E_a \cdot E_c - \vec{p}_a \cdot \vec{p}_c)$$

# The Mandelstam variables

## Useful relationships

$$
\begin{aligned}
t &= -2(E_a \cdot E_c - \vec{p}_a \cdot \vec{p}_c) \\
&= -2(\frac{\sqrt{s}}{2} \cdot \frac{\sqrt{s}}{2} - \frac{\sqrt{s}}{2} \cdot \frac{\sqrt{s}}{2} \cdot \cos\theta)
\end{aligned}
$$

massless

# The Mandelstam variables

## Useful relationships

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&= -\frac{s}{2} \cdot (1 - \cos\theta)
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massless

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\end{aligned}
$$

massless initial state and massive final state of identical particles

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massless initial state and massive final state of identical particles

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&= -\frac{s}{2} \cdot (1 - \beta \cdot \cos\theta)
\end{aligned}
$$

massless initial state and massive final state of identical particles

# Crossing relationship

Crossing relationship

$$a + b \to c + d$$

$$s = (\mathbf{a} + \mathbf{b})^2$$

$$t = (\mathbf{a} - \mathbf{c})^2$$

$$u = (\mathbf{a} - \mathbf{d})^2$$

# Crossing relationship

Crossing relationship

| $a + b \to c + d$ | $a + \bar{c} \to \bar{b} + d$ |
|---|---|
| $s = (\mathbf{a} + \mathbf{b})^2$ | $s' = (\mathbf{a} + \bar{\mathbf{c}})^2$ |
| $t = (\mathbf{a} - \mathbf{c})^2$ | $t' = (\mathbf{a} - \bar{\mathbf{b}})^2$ |
| $u = (\mathbf{a} - \mathbf{d})^2$ | $u' = (\mathbf{a} - \mathbf{d})^2$ |

# Crossing relationship

Crossing relationship

| | | |
|---|---|---|
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| $t = (\mathbf{a}-\mathbf{c})^2$ | $t' = (\mathbf{a}-\bar{\mathbf{b}})^2$ | $= (\mathbf{a}+\mathbf{b})^2$ |
| $u = (\mathbf{a}-\mathbf{d})^2$ | $u' = (\mathbf{a}-\mathbf{d})^2$ | $= (\mathbf{a}-\mathbf{d})^2$ |

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Crossing relationship

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| $a+b \to c+d$ | $a+\bar{c} \to \bar{b}+d$ | | |
| $s=(\mathbf{a}+\mathbf{b})^2$ | $s'=(\mathbf{a}+\bar{\mathbf{c}})^2$ | $=(\mathbf{a}-\mathbf{c})^2$ | $=t$ |
| $t=(\mathbf{a}-\mathbf{c})^2$ | $t'=(\mathbf{a}-\bar{\mathbf{b}})^2$ | $=(\mathbf{a}+\mathbf{b})^2$ | $=s$ |
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| $u=(\mathbf{a}-\mathbf{d})^2$ | $u'=(\mathbf{a}-\mathbf{d})^2$ | $=(\mathbf{a}-\mathbf{d})^2$ | $=u$ |

- Calculate a process as function of $s$,$t$,$u$

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Crossing relationship

$$
\begin{array}{l|llll}
a+b \to c+d & a+\bar{c} \to \bar{b}+d & & \\
s=(\mathbf{a}+\mathbf{b})^2 & s'=(\mathbf{a}+\bar{\mathbf{c}})^2 & =(\mathbf{a}-\mathbf{c})^2 & =t \\
t=(\mathbf{a}-\mathbf{c})^2 & t'=(\mathbf{a}-\bar{\mathbf{b}})^2 & =(\mathbf{a}+\mathbf{b})^2 & =s \\
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- Calculate a process as function of $s$,$t$,$u$
- Derive crossed process by $s \to t$, $t \to s$, $u \to u$

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Crossing relationship

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- We can express one process in the kinematic variables of another process (Xcheck)

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Crossing relationship

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\end{array}
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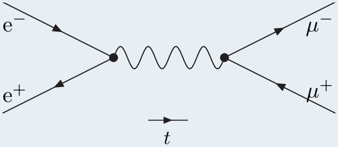
- Calculate a process as function of $s$,$t$,$u$
- Derive crossed process by $s\to t$, $t\to s$, $u\to u$
- We can express one process in the kinematic variables of another process (Xcheck)
- Global factor $-1$ for each fermion line crossed

# s channel and t channel

## s–channel: annihiliation

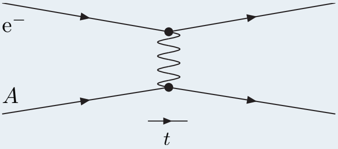
$$e^+ + e^- \to \gamma \to \mu^+ + \mu^-$$




$$\mathbf{q}_\gamma = \mathbf{p}_{e^-} + \mathbf{p}_{e^+}$$
$$s = \mathbf{q}_\gamma^2$$

## t–channel: scattering

$$e^- + A \to e^- + A$$



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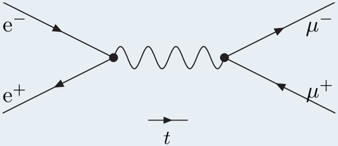


$$\begin{aligned} \mathbf{q}_\gamma &= \mathbf{p}_{e^-} + \mathbf{p}_{e^+} \\ s &= \mathbf{q}_\gamma^2 \\ (CM) &= (E_{e^-} + E_{e^+})^2 \end{aligned}$$

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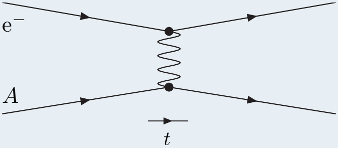

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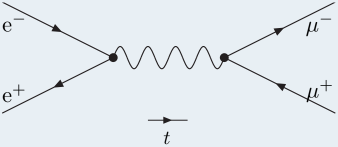


$$\begin{aligned} \mathbf{q}_\gamma &= \mathbf{p}_{e^-} + \mathbf{p}_{e^+} \\ s &= \mathbf{q}_\gamma^2 \\ (CM) &= (E_{e^-} + E_{e^+})^2 \\ &> 0 \end{aligned}$$

The photon is massive (virtual) time-like

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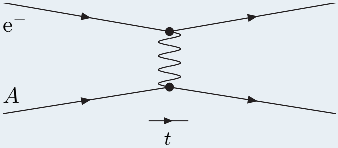

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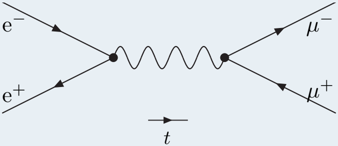


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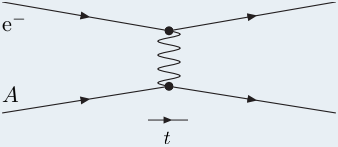


$$\mathbf{p}_{e_i^-} = \mathbf{q}_\gamma + \mathbf{p}_{e_o^-}$$

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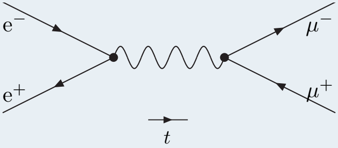
$$e^+ + e^- \to \gamma \to \mu^+ + \mu^-$$




$$\begin{aligned} \mathbf{q}_\gamma &= \mathbf{p}_{e^-} + \mathbf{p}_{e^+} \\ s &= \mathbf{q}^2_\gamma \\ (CM) &= (E_{e^-} + E_{e^+})^2 \\ &> 0 \end{aligned}$$

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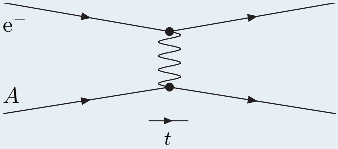
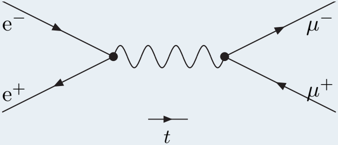



$$\begin{aligned} \mathbf{p}_{e^-_i} &= \mathbf{q}_\gamma + \mathbf{p}_{e^-_o} \\ t &= \mathbf{q}^2_\gamma \end{aligned}$$

# s channel and t channel

## s–channel: annihiliation

$$e^+ + e^- \rightarrow \gamma \rightarrow \mu^+ + \mu^-$$

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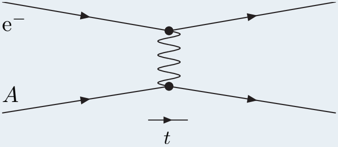


$$\begin{aligned} \mathbf{p}_{e_i^-} &= \mathbf{q}_\gamma + \mathbf{p}_{e_o^-} \\ t &= \mathbf{q}_\gamma^2 \\ &= m_e^2 + m_e^2 - 2 \cdot \mathbf{p}_{e_i^-} \cdot \mathbf{p}_{e_o^-} \end{aligned}$$

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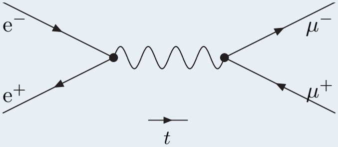


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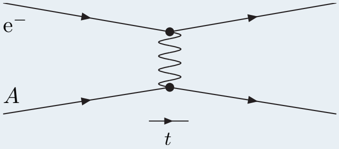


$$\begin{aligned} \mathbf{p}_{e_i^-} &= \mathbf{q}_\gamma + \mathbf{p}_{e_o^-} \\ t &= \mathbf{q}_\gamma^2 \\ &= m_e^2 + m_e^2 - 2 \cdot \mathbf{p}_{e_i^-} \cdot \mathbf{p}_{e_o^-} \\ &\approx -2(E_i E_o - |\vec{p}_i||\vec{p}_o| \cos\theta) \end{aligned}$$

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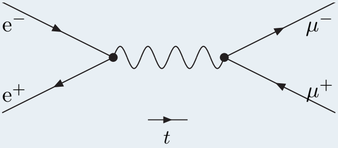


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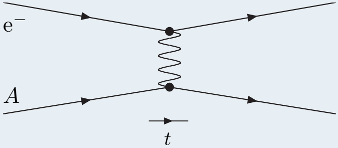


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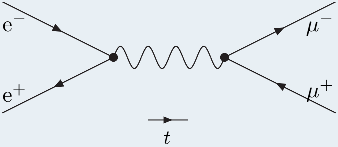
# s channel and t channel

## s–channel: annihiliation

$$e^+ + e^- \to \gamma \to \mu^+ + \mu^-$$

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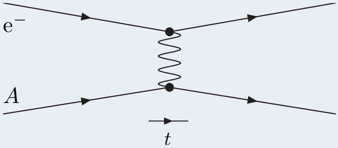


$$\begin{aligned} \mathbf{p}_{e_i^-} &= \mathbf{q}_\gamma + \mathbf{p}_{e_o^-} \\ t &= \mathbf{q}_\gamma^2 \\ &= m_e^2 + m_e^2 - 2 \cdot \mathbf{p}_{e_i^-} \cdot \mathbf{p}_{e_o^-} \\ &\approx -2(E_i E_o - |\vec{p}_i||\vec{p}_o| \cos\theta) \\ &\approx -2E_i E_o(1 - \cos\theta) \\ &\leq 0 \end{aligned}$$

The photon is massive space-like

# Cross section and total width

## Cross Section

- The cross section $\sigma$ is the ratio of the transition rate and the flux of incoming particles.
- Its unit is $\mathrm{cm}^2$
- $1\mathrm{b} = 10^{-24}\mathrm{cm}^2$ (puts barn in perspective, doesn't it?)

Two ingredients:

- the interaction transforming initial state $|i\rangle$ to a final state $\langle f|$ of $m$ particles with four-vectors $\mathbf{p}'_\mathbf{i}$

$$\mathrm{d}\sigma = \qquad |\mathcal{M}|^2$$

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Two ingredients:

- the interaction transforming initial state $|i\rangle$ to a final state $\langle f|$ of $m$ particles with four-vectors $\mathbf{p'_i}$
- kinematics (including Lorentz-Invariant phase space element)

$$\mathrm{d}\sigma = \frac{1}{2S_{12}} \prod_{i=1}^{m} \frac{\mathrm{d}^3 p'_i}{(2\pi)^3 2E'^0_i} (2\pi)^4 \delta(\mathbf{p'_1} + \ldots + \mathbf{p'_m} - \mathbf{p_1} - \mathbf{p_2}) |\mathcal{M}|^2$$

with (originating from flux) $S_{12} = \sqrt{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}$

# Cross section and total width

## Total Width or Decay Rate

- Total width is the inverse of the lifetime of the particle
- unit: energy, e.g., GeV.
- Closely related, but not identical to the cross section

$$\mathrm{d}\Gamma = \frac{1}{2E}\prod_{i=1}^{m}\frac{\mathrm{d}^3 p_i'}{(2\pi)^3 2E_i'^0}(2\pi)^4\delta(\mathbf{p_1'}+\ldots+\mathbf{p_m'}-\mathbf{p_1})|\mathcal{M}|^2$$

For the decay of an unpolarized particle of mass $M$ into two particles (in the CM frame $\vec{p}_1' = -\vec{p}_2'$):

$$\mathrm{d}\Gamma = \frac{1}{32\pi^2}\frac{|\vec{p}_1'|}{M^2}|\mathcal{M}|^2\mathrm{d}\Omega$$

where $\Omega$ is the solid angle with $\mathrm{d}\Omega = \mathrm{d}\phi\mathrm{d}\cos\theta$

# Cross section and total width

## Cross section and total width for a final state with 2 particles

Cross section $2 \to 2$ reaction with four massless particles:

$$d\sigma = \frac{1}{64\pi^2} \frac{|\mathcal{M}|^2}{s} d\Omega$$

Width of a massive particle ($\sqrt{s} = M$) decaying to two massless particles in the final state $|\vec{p}_1'| = \sqrt{s}/2$:

$$d\Gamma = \frac{1}{64\pi^2} \frac{|\mathcal{M}|^2}{\sqrt{s}} d\Omega$$

Study of the phase space in Problem Solving with applications to 2-body.

# Description of an unstable particle

- Particles: plane waves
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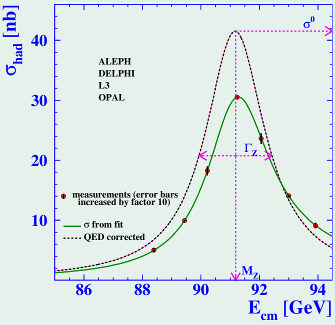


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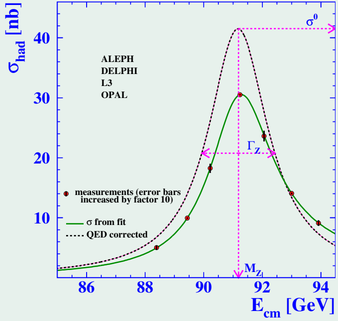


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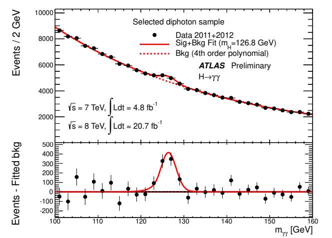


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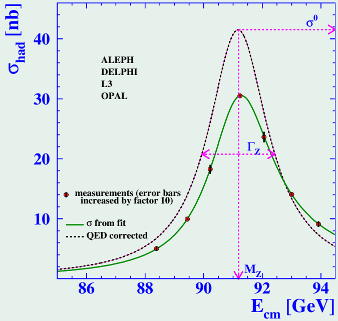


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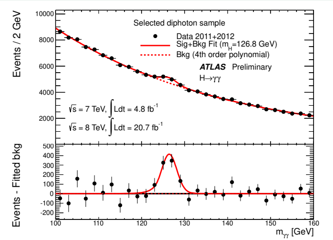


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The experimental resolution is the origin (error propagation):

$$m_H = \sqrt{(\mathbf{p_1}^\gamma + \mathbf{p_2}^\gamma)^2} = \sqrt{2E_1^\gamma E_2^\gamma (1 - \cos\theta)}$$

# Description of an unstable particle

Suppose that we have two (and exactly two) possible decays for the particle $a$:

$$\begin{array}{rcl} a & \rightarrow & b + c \\ a & \rightarrow & d + e \end{array}$$

then:

$$\Gamma = \Gamma_{bc} + \Gamma_{de}$$

If a particle of a given mass can decay to more final states than another one with the same mass, it will have a shorter lifetime

## Branching ratio

$\mathcal{B}(\mathrm{a} \rightarrow \mathrm{b} + \mathrm{c}) = \Gamma_{bc}/\Gamma$

The branching ratio: Of $N$ decays of particle $a$, a fraction $\mathcal{B}$ will be the final state with the particles $b$ and $c$. $\Gamma_{bc}$ is a partial width of particle $a$.

Remember: for the calculation $\Gamma$ ALL final states (partial widths) have to be considered.

# Introduction

## Why do we need quantum field theory ? $E = mc^2$ and QFT

### from $E = mc^2$ to quantum field theory

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This means that if there is enough energy, we can create a particle with a given mass $m$. However, due to conservation laws, it will most probably be necessary to produce twice the particle's mass (particle and antiparticle).
Hence

- Particle number is not fixed
- The types of particles present is not fixed

This is in direct conflict with nonrelativistic quantum mechanics and for example the Schrödinger equation that treats a constant number of particles of a certain type.

# Special relativity and quantum field theory

## Attempts to incorporate special relativity in Quantum mechanics

### Quantum mechanics and special relativity

Schrödinger equation contained first order time derivative and second order space derivatives

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

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First attempt consisted to promote the time derivative to the second order. This resulted in the Klein Gordon equation :

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} = \frac{m^2c^2}{\hbar^2}\psi$$

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But this leads to funny features:

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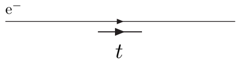
Dirac solved the problems by reducing the spatial derivative power.
Resulted in the Dirac equation.

# Evaluate a process

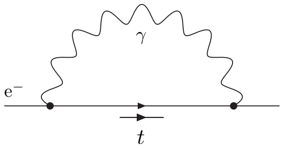
## Diagram orders: LO




- Leading Order (LO) diagram is the simplest diagram
- The electron is on-shell ($\mathbf{p}^2 = m_e^2$), no interactions

# Evaluate a process

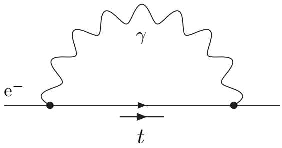
## Diagram orders... NLO




- NLO (next-to-leading order) diagram
- Process not allowed in classical mechanics
- Heisenberg: $\Delta E \Delta t \geq 1 \rightarrow$ process allowed for reabsorption after $\Delta t \sim 1/\Delta E$

# Evaluate a process

## Diagram orders... NLO




- Quantum mechanics: add all diagrams, but that would also include $N_\gamma = \infty$
- Each vertex is an interaction and each interaction has a strength ($|\mathcal{M}|^2 \sim \alpha = 1/137$)
- Perturbation theory with Sommerfeld convergence

# Feynman calculation

## Rough recipe

### Process calculation

- Construct the Lagrangian of Free Fields
- Introduce interactions via the minimal substitution scheme
- Derive Feynman rules
- Construct (ALL) Feynman diagrams of the process
- Apply Feynman rules

Some aspects are not part of these lectures, but will sketch the ideas

# The Lagrangian

## What we want to describe

- Remember the particle zoo

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \begin{pmatrix} c_L \\ s_L \end{pmatrix} \quad \begin{pmatrix} t_L \\ b_L \end{pmatrix}$$

$$\begin{pmatrix} \nu_{e_L} \\ e_L \end{pmatrix} \quad \begin{pmatrix} \nu_{\mu_L} \\ \mu_L \end{pmatrix} \quad \begin{pmatrix} \nu_{\tau_L} \\ \tau_L \end{pmatrix}$$

$$\begin{matrix} u_R & c_R & t_R \\ d_R & s_R & b_R \\ e_R & \mu_R & \tau_R \end{matrix}$$

$$\begin{matrix} \gamma \\ g \\ W^{\pm}, Z^{\circ} \\ H \end{matrix}$$

# The Lagrangian

## What we want to describe

- Remember the particle zoo
- treat only the carrier of the interaction $\gamma$
- as well as the $\mathrm{e}$

$$\begin{pmatrix} \mathrm{e_L} \end{pmatrix}$$

$$\mathrm{e_R}$$

$$\gamma$$

# Lagrangian field theory

## The Lagrangian and the Action

The Lagrangian is defined by

$$L = T - V$$

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The action is the time integration of the Lagrangian, $S = \int L dt$. This is a functional: its argument is a function and it returns a number.

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## The Lagrangian and the Action

The Lagrangian is defined by

$$L = T - V$$

The action is the time integration of the Lagrangian, $S = \int L dt$. This is a functional: its argument is a function and it returns a number.
Assuming that the Action should be minimal

$$S = \int_{t_1}^{t_2} L(q, \dot{q}) dt \text{ with } \delta S = 0$$

(the $q_i(t)$ being the generalized coordinates)
leads to the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

The familiar equations of motion can be obtained from this equation.

# Lagrangian field theory

## Lagrangian density

Lagrangian formalism is now applied to fields, which are functions of spacetime $\psi(x, t)$. The Lagrangian is, in the continuous case, the space integration of the Lagrangian density.

$$L = T - V = \int \mathcal{L} d^3 x$$

and the action becomes

$$S = \int L dt = \int \mathcal{L} d^4 x$$

Typically,

$$\mathcal{L} = \mathcal{L}(\psi, \partial_\mu \psi)$$

From a Lagrangian density and the Euler-Lagrange equation, equations governing the evolution of particles (i.e. fields) can be derived.

# Quantum Electrodynamics

## The photon

Maxwell equations:

$$\begin{aligned}\partial_\mu F^{\mu\nu}(\mathbf{x}) &= j^\nu(\mathbf{x})\\ \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}(\mathbf{x}) &= 0\end{aligned}$$

with the photon field tensor:

$$\begin{aligned}F^{\mu\nu}(\mathbf{x}) &= \partial^\mu A^\nu(\mathbf{x}) - \partial^\nu A^\mu(\mathbf{x})\\ F_{\mu\nu}(\mathbf{x}) &= \partial_\mu A_\nu(\mathbf{x}) - \partial_\nu A_\mu(\mathbf{x})\end{aligned}$$

$A^\mu$ being the usual vector potential,

$\mathbf{A} = (\psi, \vec{A})$ and $j^\nu(\mathbf{x}) = \begin{pmatrix} \rho(\mathbf{x}) \\ \vec{j}(\mathbf{x}) \end{pmatrix}$,

the current density.

$F$ is antisymmetric, $F_{\mu\nu} = -F_{\nu\mu}$ and has 6 independent components

$$\vec{E} = (E_x, E_y, E_z) = (F_{01}, F_{02}, F_{03})$$

$$\vec{B} = (B_x, B_y, B_z) = (-F_{23}, -F_{31}, -F_{12})$$

The potentials are not unique and are determined up to a *gauge* transformation $A_\mu \to A_\mu + \partial_\mu\chi$ where $\chi$ is arbitrary.

The field strengths are invariant under this transformation (the $\partial$s commute...).

We can restrict the gauge and there exist plenty of them. We will often refer to the Lorentz gauge $\partial_\mu A^\mu = 0$.

The current satisfies the continuity equation which, in 4-dimension, is a vanishing 4-divergence $\partial_\mu j^\mu = 0$.

If you use the Lorentz gauge on the inhomogeneous Maxwell equations, you get the simplified and equivalent equation

$$\Box A^\nu = j^\nu$$

# Quantum Electrodynamics

## The fermions

Schrödinger equation is $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$, with $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V$.

However $\hat{H}$ should be chosen now to respect special relativy.

Writing $P^\mu = i\partial^\mu$, $\hat{H}$ should fulfill

$$\hat{H}^2 = \vec{P}^2 + m^2 = \sum_k (P^k)^2 + m^2$$

Let's take a matrix form to solve the problem (obviously it cannot be a number...)

$$\hat{H} = \sum_{k=1}^{3} \alpha^k P^k + \beta m = \vec{\alpha}.\vec{P} + \beta m$$

By putting $\hat{H}$ to the square you get some constraints on $\vec{\alpha}$ and $\beta$:

$$\alpha^k\alpha^l + \alpha^l\alpha^k = 2\delta^{kl},$$
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This implies that : $\beta^2 = 1$, $(\alpha^k)^2 = 1$, $Tr(\beta) = Tr(\alpha) = 1$, and eigenvalues of $\alpha^k$ and $\beta$ are $\pm 1$

The wave function has to be a 4-component column.

The $\alpha$ and $\beta$ can be redefined in $\gamma^0 = \beta$, $\gamma^k = \beta\alpha^k$, with $\gamma^\mu = (\gamma^0, \vec{\gamma})$

One common representation is the *Dirac representation* :

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

The previous set of equations is rewritten as $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$

$\gamma^5$ is often introduced, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, leading to

$$(\gamma^5)^2 = \mathbb{1}_4 \quad \text{and} \quad \{\gamma^5, \gamma^\mu\} = 0$$

# Quantum Electrodynamics

## Dirac equation

The Dirac equation is then $i\frac{\partial\psi}{\partial t} = (\vec{\alpha}.\vec{P} + \beta m)\psi$

and by multiplication with $\beta$, we obtain $i\beta\partial_0\psi = -i\beta\alpha^k\partial_k\psi + \beta^2 m\psi$

which transforms into $i\gamma^0\partial_0\psi + i\gamma^k\partial_k\psi - m\psi = i\gamma^\mu\partial_\mu\psi - m\psi = 0$

The Dirac equation final form is:

$$(i\gamma^\mu\partial_\mu - m)\psi(\mathbf{x}) = 0$$

## Lagrangians...

Applying Euler-Lagrange on the Lagrangian $\mathcal{L} = \bar{\psi}(\mathbf{x})(i\gamma^\mu\partial_\mu - m)\psi(\mathbf{x})$ gives the Dirac equation with $\bar{\psi} = \psi^\dagger\gamma^0 = \psi^{\mathrm{T}^\star}\gamma^0$

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## The free Lagrangian ($\mathcal{L}_0$)

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}(\mathbf{x})F^{\mu\nu}(\mathbf{x}) + \bar{\psi}(\mathbf{x})(i\gamma^\mu\partial_\mu - m)\psi(\mathbf{x})$$

# Quantum Electrodynamics

## Minimal Substitution

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$$i\partial_\mu \to i\partial_\mu + eA_\mu(\mathbf{x})$$

$$\bar{\psi}(\mathbf{x})\gamma^\mu i\partial_\mu \psi(\mathbf{x})$$

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### Interaction Lagrangian $\mathcal{L}'$

$$\mathcal{L}' = -j^\mu A_\mu = e\bar{\psi}(\mathbf{x})\gamma^\mu A_\mu(\mathbf{x})\psi(\mathbf{x})$$

the negative sign for $j^\mu = -e\bar{\psi}(\mathbf{x})\gamma^\mu\psi(\mathbf{x})$

# Quantum Electrodynamics

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### Dirac equation for adjoint spinor

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Dirac equation

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### Dirac equation for adjoint spinor

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\begin{aligned}
i\gamma^\mu\partial_\mu\psi - m\psi &= 0\\
-i(\gamma^\mu)^\star\partial_\mu\psi^\star - m\psi^\star &= 0\\
-i(\partial_\mu\psi^\dagger)(\gamma^\mu)^\dagger - m\psi^\dagger &= 0\\
-i(\partial_\mu\psi^\dagger)\gamma^\circ\gamma^\circ(\gamma^\mu)^\dagger\gamma^\circ - m\psi^\dagger\gamma^\circ &= 0\\
-i(\partial_\mu\bar{\psi})\gamma^\circ(\gamma^\mu)^\dagger\gamma^\circ - m\bar{\psi} &= 0\\
-i(\partial_\mu\bar{\psi})\gamma^\mu - m\bar{\psi} &= 0\\
i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} &= 0
\end{aligned}
$$

# Quantum Electrodynamics

## Dirac equation

### Dirac equation for adjoint spinor

$$\begin{aligned}
i\gamma^\mu\partial_\mu\psi - m\psi &= 0\\
-i(\gamma^\mu)^\star\partial_\mu\psi^\star - m\psi^\star &= 0\\
-i(\partial_\mu\psi^\dagger)(\gamma^\mu)^\dagger - m\psi^\dagger &= 0\\
-i(\partial_\mu\psi^\dagger)\gamma^\circ\gamma^\circ(\gamma^\mu)^\dagger\gamma^\circ - m\psi^\dagger\gamma^\circ &= 0\\
-i(\partial_\mu\bar{\psi})\gamma^\circ(\gamma^\mu)^\dagger\gamma^\circ - m\bar{\psi} &= 0\\
-i(\partial_\mu\bar{\psi})\gamma^\mu - m\bar{\psi} &= 0\\
i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} &= 0
\end{aligned}$$

### EM current conserved

$$\begin{aligned}
\partial_\mu j^\mu &= \partial_\mu[-e\bar{\psi}\gamma^\mu\psi]\\
&= -e(\partial_\mu\bar{\psi})\gamma^\mu\psi - e\bar{\psi}\gamma^\mu\partial_\mu\psi && \textit{Dirac}\\
&= -ime\bar{\psi}\psi + iem\bar{\psi}\psi && \textit{Dirac adjoint}\\
&= 0
\end{aligned}$$

# Gauge invariance

## Gauge Invariance

Invariance of the Lagrangian under local $U(1)$ transformations
or: why should physics depend on the location ?

$$\begin{array}{lcl} A_\mu & \rightarrow & A_\mu + \partial_\mu \Lambda(\mathbf{x}) \\ \psi(\mathbf{x}) & \rightarrow & \exp\left(ie\Lambda(\mathbf{x})\right)\psi(\mathbf{x}) \end{array}$$

$$\mathcal{L}_0 + \mathcal{L}' = \mathcal{L} \rightarrow \mathcal{L}$$

Local gauge invariance under a $U(1)$ gauge symmetry (1929 Weyl)
if $\Lambda \neq f(\mathbf{x})$ it is a global $U(1)$ symmetry.

# Gauge invariance

$U(1)$ Gauge invariance:

## Photon field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

# Gauge invariance

$U(1)$ Gauge invariance:

## Photon field

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \partial_\mu(A_\nu + \partial_\nu \Lambda) - \partial_\nu(A_\mu + \partial_\mu \Lambda) \end{aligned}$$

# Gauge invariance

$U(1)$ Gauge invariance:

## Photon field

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \partial_\mu (A_\nu + \partial_\nu \Lambda) - \partial_\nu (A_\mu + \partial_\mu \Lambda) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda \end{aligned}$$

# Gauge invariance

$U(1)$ Gauge invariance:

## Photon field

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= \partial_\mu(A_\nu + \partial_\nu \Lambda) - \partial_\nu(A_\mu + \partial_\mu \Lambda) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda \quad \partial_\mu \partial_\nu = \partial_\nu \partial_\mu
\end{aligned}
$$

# Gauge invariance

$U(1)$ Gauge invariance:

## Photon field

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= \partial_\mu(A_\nu + \partial_\nu \Lambda) - \partial_\nu(A_\mu + \partial_\mu \Lambda) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda \quad \partial_\mu \partial_\nu = \partial_\nu \partial_\mu \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu
\end{aligned}$$

# Gauge invariance

$U(1)$ Gauge invariance:

## Photon field

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F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= \partial_\mu (A_\nu + \partial_\nu \Lambda) - \partial_\nu (A_\mu + \partial_\mu \Lambda) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda \quad \partial_\mu \partial_\nu = \partial_\nu \partial_\mu \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= F_{\mu\nu}
\end{aligned}
$$

Photon field ok

# Gauge invariance

## Fermion field

$$\bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi$$

# Gauge invariance

## Fermion field

$$
\begin{aligned}
& \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \\
= \quad & \psi^\dagger \gamma^0 (i\gamma^\mu \partial_\mu - m)\psi
\end{aligned}
$$

# Gauge invariance

## Fermion field

$$
\begin{aligned}
& \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \\
= \quad & \psi^\dagger \gamma^0 (i\gamma^\mu \partial_\mu - m)\psi \\
\to \quad & \psi^\dagger \exp(-ie\Lambda)\gamma^0 (i\gamma^\mu \partial_\mu - m)(\psi \exp(ie\Lambda))
\end{aligned}
$$

# Gauge invariance

## Fermion field

$$
\begin{aligned}
& \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \\
\rightarrow \quad & \psi^\dagger \exp(-ie\Lambda)\gamma^0(i\gamma^\mu\partial_\mu - m)(\psi \exp(ie\Lambda)) \\
= \quad & \exp(-ie\Lambda)\bar{\psi}(i\gamma^\mu\partial_\mu - m)(\psi \exp(ie\Lambda))
\end{aligned}
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# Gauge invariance

## Fermion field

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\begin{aligned}
& \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \\
\rightarrow\quad & \psi^\dagger \exp(-ie\Lambda)\gamma^0(i\gamma^\mu\partial_\mu - m)(\psi \exp(ie\Lambda)) \\
=\quad & \exp(-ie\Lambda)\bar{\psi}(i\gamma^\mu\partial_\mu - m)(\psi \exp(ie\Lambda)) \\
=\quad & \exp(-ie\Lambda)\bar{\psi} i\gamma^\mu(\partial_\mu\psi)\exp(ie\Lambda) \\
+\quad & \exp(-ie\Lambda)\bar{\psi} i\gamma^\mu\psi\partial_\mu \exp(ie\Lambda)) \\
+\quad & \exp(-ie\Lambda)\bar{\psi}(-m)\psi \exp(ie\Lambda)
\end{aligned}
$$

# Gauge invariance

## Fermion field

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\begin{aligned}
& \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \\
\rightarrow\ & \psi^\dagger \exp(-ie\Lambda)\gamma^0(i\gamma^\mu\partial_\mu - m)(\psi \exp(ie\Lambda)) \\
=\ & \exp(-ie\Lambda)\bar{\psi}(i\gamma^\mu\partial_\mu - m)(\psi \exp(ie\Lambda)) \\
=\ & \exp(-ie\Lambda)\bar{\psi}i\gamma^\mu(\partial_\mu\psi)\exp(ie\Lambda) \\
+\ & \exp(-ie\Lambda)\bar{\psi}i\gamma^\mu\psi\partial_\mu \exp(ie\Lambda)) \\
+\ & \exp(-ie\Lambda)\bar{\psi}(-m)\psi \exp(ie\Lambda) \\
=\ & \bar{\psi}i\gamma^\mu(\partial_\mu\psi)
\end{aligned}
$$

# Gauge invariance

## Fermion field

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\begin{aligned}
& \bar\psi(i\gamma^\mu\partial_\mu - m)\psi \\
\rightarrow\ & \psi^\dagger \exp(-ie\Lambda)\gamma^0(i\gamma^\mu\partial_\mu - m)(\psi\exp(ie\Lambda)) \\
=\ & \exp(-ie\Lambda)\bar\psi(i\gamma^\mu\partial_\mu - m)(\psi\exp(ie\Lambda)) \\
=\ & \exp(-ie\Lambda)\bar\psi i\gamma^\mu(\partial_\mu\psi)\exp(ie\Lambda) \\
+\ & \exp(-ie\Lambda)\bar\psi i\gamma^\mu\psi\partial_\mu\exp(ie\Lambda)) \\
+\ & \exp(-ie\Lambda)\bar\psi(-m)\psi\exp(ie\Lambda) \\
=\ & \bar\psi i\gamma^\mu(\partial_\mu\psi) + \exp(-ie\Lambda)\bar\psi i\gamma^\mu\psi ie\partial_\mu\Lambda\exp(ie\Lambda)
\end{aligned}
$$

# Gauge invariance

## Fermion field

$$
\begin{aligned}
& \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \\
\rightarrow \quad & \psi^\dagger \exp(-ie\Lambda)\gamma^0(i\gamma^\mu\partial_\mu - m)(\psi\exp(ie\Lambda)) \\
= \quad & \exp(-ie\Lambda)\bar{\psi}(i\gamma^\mu\partial_\mu - m)(\psi\exp(ie\Lambda)) \\
= \quad & \exp(-ie\Lambda)\bar{\psi}i\gamma^\mu(\partial_\mu\psi)\exp(ie\Lambda) \\
+ \quad & \exp(-ie\Lambda)\bar{\psi}i\gamma^\mu\psi\partial_\mu\exp(ie\Lambda)) \\
+ \quad & \exp(-ie\Lambda)\bar{\psi}(-m)\psi\exp(ie\Lambda) \\
= \quad & \bar{\psi}i\gamma^\mu(\partial_\mu\psi) + \exp(-ie\Lambda)\bar{\psi}i\gamma^\mu\psi ie\partial_\mu\Lambda\exp(ie\Lambda) \\
+ \quad & \bar{\psi}(-m)\psi
\end{aligned}
$$

# Gauge invariance

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\begin{aligned}
& \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \\
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=\quad & \exp(-ie\Lambda)\bar{\psi}i\gamma^\mu(\partial_\mu \psi)\exp(ie\Lambda) \\
+\quad & \exp(-ie\Lambda)\bar{\psi}i\gamma^\mu \psi \partial_\mu \exp(ie\Lambda)) \\
+\quad & \exp(-ie\Lambda)\bar{\psi}(-m)\psi \exp(ie\Lambda) \\
=\quad & \bar{\psi}i\gamma^\mu(\partial_\mu \psi) - e\bar{\psi}\gamma^\mu(\partial_\mu \Lambda)\psi \\
+\quad & \bar{\psi}(-m)\psi
\end{aligned}
$$

# Gauge invariance

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& \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \\
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=\ & \exp(-ie\Lambda)\bar{\psi} i\gamma^\mu(\partial_\mu \psi)\exp(ie\Lambda) \\
+\ & \exp(-ie\Lambda)\bar{\psi} i\gamma^\mu \psi \partial_\mu \exp(ie\Lambda)) \\
+\ & \exp(-ie\Lambda)\bar{\psi}(-m)\psi \exp(ie\Lambda) \\
=\ & \bar{\psi} i\gamma^\mu(\partial_\mu \psi) - e\bar{\psi}\gamma^\mu(\partial_\mu \Lambda)\psi \\
+\ & \bar{\psi}(-m)\psi \\
=\ & \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu(\partial_\mu \Lambda)\psi
\end{aligned}$$

# Gauge invariance

## Interaction

$$e\bar{\psi}\gamma^{\mu}A_{\mu}\psi(\mathbf{x})$$

# Gauge invariance

## Interaction

$$
\begin{aligned}
& e\bar{\psi}\gamma^{\mu}A_{\mu}\psi(\mathbf{x}) \\
= \quad & e\exp\left(-ie\Lambda\right)\bar{\psi}\gamma^{\mu}(A_{\mu}+\partial_{\mu}\Lambda)\psi\exp\left(ie\Lambda\right)
\end{aligned}
$$

# Gauge invariance

## Interaction

$$
\begin{aligned}
& e\bar{\psi}\gamma^\mu A_\mu \psi(\mathbf{x}) \\
= \quad & e \exp\left(-ie\Lambda\right)\bar{\psi}\gamma^\mu (A_\mu + \partial_\mu \Lambda)\psi \exp\left(ie\Lambda\right) \\
= \quad & e\bar{\psi}\gamma^\mu (A_\mu + \partial_\mu \Lambda)\psi
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& e\bar{\psi}\gamma^\mu A_\mu \psi(\mathbf{x}) \\
= \quad & e \exp(-ie\Lambda)\bar{\psi}\gamma^\mu(A_\mu + \partial_\mu\Lambda)\psi \exp(ie\Lambda) \\
= \quad & e\bar{\psi}\gamma^\mu(A_\mu + \partial_\mu\Lambda)\psi \\
= \quad & e\bar{\psi}\gamma^\mu A_\mu\psi + e\bar{\psi}\gamma^\mu(\partial_\mu\Lambda)\psi
\end{aligned}
$$

# Gauge invariance

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\begin{aligned}
& e\bar{\psi}\gamma^\mu A_\mu \psi(\mathbf{x}) \\
= \quad & e \exp(-ie\Lambda)\bar{\psi}\gamma^\mu (A_\mu + \partial_\mu \Lambda)\psi \exp(ie\Lambda) \\
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= \quad & e\bar{\psi}\gamma^\mu A_\mu \psi + e\bar{\psi}\gamma^\mu (\partial_\mu \Lambda)\psi
\end{aligned}
$$

- Interaction term combined with fermion field ($-ie\bar{\psi}\gamma^\mu \partial_\mu \Lambda \psi$) ok

# Gauge invariance

## Interaction

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\begin{aligned}
& e\bar{\psi}\gamma^\mu A_\mu \psi(\mathbf{x}) \\
= \quad & e \exp(-ie\Lambda)\bar{\psi}\gamma^\mu(A_\mu + \partial_\mu\Lambda)\psi \exp(ie\Lambda) \\
= \quad & e\bar{\psi}\gamma^\mu(A_\mu + \partial_\mu\Lambda)\psi \\
= \quad & e\bar{\psi}\gamma^\mu A_\mu \psi + e\bar{\psi}\gamma^\mu(\partial_\mu\Lambda)\psi
\end{aligned}
$$

- Interaction term combined with fermion field ($-ie\bar{\psi}\gamma^\mu\partial_\mu\Lambda\psi$) ok
- gauge invariance of the fermion field cries for the introduction of a gauge boson!

# Free plane wave solution

## Solutions to the Dirac equation

The free-particle plane wave solution is the most natural solution to the Dirac equation

$$\psi(\mathbf{x}) = u(E,p)e^{i(p.x-Et)}$$

$u(E,p)$ being a four-component Dirac spinor.
The overall wavefunction must satisfy the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

From the form of this solution and of the Dirac equation, the Dirac Spinor should satisfy

$$(\gamma^\mu p_\mu - m)u = 0$$

# Free plane wave solution

## Negative energy solutions are back

If we look at those solutions at rest, we get

$$\psi(E, 0) = u(E, 0)e^{-iEt}$$

That reduces to the Spinor equation $E\gamma^0 u = mu$ that can be expressed as

$$E\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = m\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

Out of the four orthogonal solutions, 2 have positive energies ($E = m$) and two have *negative ones* ($E = -m$). Hence, we did not resolve the energy solution problem !

It can be shown that those states are also eigenstates of the $\hat{S}_z$ operator, two representing spin-up and two spin-down solutions.

# Stückelberg - Feynman

## Stückelberg - Feynman

The *negative* energy solutions are interpreted as *particles* propagating *backwards* in time.

They correspond to *positive* energy *anti-particle* propagating *forwards* in time.

# Stückelberg - Feynman

## Anti-particle spinors

The wavefunction of the particle (its exponential part $e^{-iEt}$) is unchanged under the simultaneous transformation of energy and time $E \to -E$, $t \to -t$.

Instead of working with negative energy solutions moving backwards in time $u_3$ and $u_4$, it is simpler to change the definition and work with spinors for anti-particles moving forward in time, $v_1$ and $v_2$, reversing the sign of $E$ and $p$ :

$$v_1(E,p)e^{-i(p.x-Et)} = u_4(-E,-p)e^{i[-p.x-(-Et)]}$$

$$v_2(E,p)e^{-i(p.x-Et)} = u_3(-E,-p)e^{i[-p.x-(-Et)]}$$

which comes originally from identifying the anti-particle spinors as

$$\psi(\mathbf{x}) = v(E,p)e^{-i(p.x-Et)}$$

where the exponent sign as been reversed. Once more we have two negative energy anti-particle solutions : $v_3$ and $v_4$. We could work either with particle spinors ($u_i$) or anti-particle spinors ($v_i$).

We prefer sometimes to mix particles and anti-particles to have only *positive* energy solutions to the Dirac equations.

# Stückelberg - Feynman

## Charge conjugaison and Dirac equation

The Dirac equation for electrons, incorporating the interaction (minimal substitution) is

$$\gamma^\mu(\partial_\mu - ieA_\mu)\psi + im\psi = 0$$

Let's try to get the equation for a positron. First we multiply on the left by $-i\gamma^2$ the complex conjugate:

$$-i\gamma^2(\gamma^\mu)^\star(\partial_\mu + ieA_\mu)\psi^\star - m\gamma^2\psi^\star = 0$$

which becomes using $\gamma$ relations

$$\gamma^\mu(\partial_\mu + ieA_\mu)i\gamma^2\psi^\star + imi\gamma^2\psi^\star = 0$$

Defining $\psi\prime$ as $\psi' = i\gamma^2\psi^\star$, the Dirac equation becomes

$$\gamma^\mu(\partial_\mu + ieA_\mu)\psi' + im\psi' = 0$$

We obtained the Dirac equation for a positron if we identify it with $\psi' = \hat{C}\psi = i\gamma^2\psi$, $\hat{C}$ being then the charge conjugaison operator.

# Stückelberg - Feynman

## $\hat{C}$ effect on the spinors

From the Dirac equation, the $u_1$, for example, can be found: $u_1(E,p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}$ and

applying $\hat{C}$ on $u_1$, we get,

$$i\gamma^2 u_i^\star = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}^\star = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

which can be identified as the solution $v_1$ that we mentioned earlier.

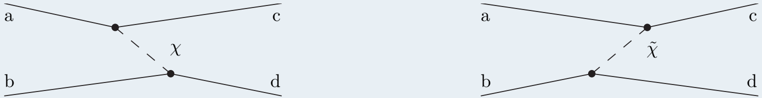
$$\psi = u_1 e^{i(p.x-Et)} \rightarrow^{(\hat{C})} \psi' = v_1 e^{-i(p.x-Et)}$$

$$\psi = u_2 e^{i(p.x-Et)} \rightarrow)^{\hat{C})} \psi' = v_2 e^{-i(p.x-Et)}$$

# Time ordered perturbation theory

## The process

We suppose that two particles $(a, b)$ interact and produce $(c, d)$ in the final state with the exchange of a particle $\chi$. This can be done with two time ordered pictures :




The initial and final states are the same, but the intermediate one is $(b, c, \chi)$ or $(a, \tilde{\chi}, d)$.

## Perturbation Theory

The matrix elements are in perturbation theory,

$$T_{fi}^{ab} = \frac{< f|V|j >< j|V|i >}{E_i - E_j} = \frac{< d|V|\chi + b >< c + \chi|V|a >}{(E_a + E_b) - (E_c + E_\chi + E_b)}$$

This is the non-invariant matrix-element that we normalize by $\sqrt{\prod_k 2E_k}$ (k is the index of the particles involved) in order to make it Lorentz invariant

$$\mathcal{M}_{a\to c+\chi} =< c + \chi|V|a > \sqrt{2E_a 2E_c 2E_\chi}$$

Remember that by changing the Lorentz frame, the volume contracts by a factor $\gamma = E/m$. Hence, the density increases by $\gamma$. The convention consists in normalizing the wave functions to $2E$ particles/unit volume $\int \psi'^\star \psi' dV = 2E$, instead of $\int \psi^\star \psi dV = 1$ in NR-QM. Hence, $\psi' = \sqrt{2E}\psi$.

# Time ordered perturbation theory

The requirement that the matrix elements $\mathcal{M}_{a\to c+\chi}$ are Lorentz invariant is a strong constraint on their form. Let's take the example of the scalar coupling $\mathcal{M}_{a\to c+\chi} = g_a$. Hence,

$$< c + \chi|V|a >= \frac{g_a}{\sqrt{2E_a 2E_c 2E_\chi}}$$

Similarly

$$< d|V|\chi + b >= \frac{g_b}{\sqrt{2E_b 2E_d 2E_\chi}}$$

where $g_b$ is the coupling of $b$ with $(\chi, d)$. $T_{fi}^{ab}$ becomes

$$T_{fi}^{ab} = \frac{< d|V|\chi + b >< c + \chi|V|a >}{(E_a + E_b) - (E_c + E_\chi + E_b)} = \frac{1}{2E_\chi} \frac{1}{\sqrt{2E_a 2E_b 2E_c 2E_d}} \frac{g_a g_b}{(E_a - E_c - E_\chi)}$$

Getting back to the Lorentz invariant matrix elements expressed in terms of the properly normalized wave-functions $\mathcal{M}_{fi}^{ab} = \sqrt{2E_a 2E_b 2E_c 2E_d}\, T_{fi}^{ab}$ so that

$$\mathcal{M}_{fi}^{ab} = \frac{1}{2E_\chi} \frac{g_a g_b}{(E_a - E_c - E_\chi)}$$

The second diagram would give $\mathcal{M}_{fi}^{ba} = \frac{1}{2E_\chi} \frac{g_a g_b}{(E_b - E_d - E_\chi)}$

# Time ordered perturbation theory

## Sum of the amplitudes

$$\mathcal{M}_{fi} = \mathcal{M}_{fi}^{ab} + \mathcal{M}_{fi}^{ba}$$

gives

$$\mathcal{M}_{fi} = \frac{g_a g_b}{2E_\chi}\left(\frac{1}{E_b - E_d - E_\chi} + \frac{1}{E_a - E_c - E_\chi}\right) = \frac{g_a g_b}{(E_a - E_c)^2 - E_\chi^2}$$

considering that $E_b - E_d = E_c - E_a$.

Indeed, $\mathbf{p}_\chi = (\mathbf{p}_b - \mathbf{p}_d) = -(\mathbf{p}_a - \mathbf{p}_c)$ so that $E_\chi^2 = p_\chi^2 + m_\chi^2 = (p_a - p_c)^2 - m_\chi^2$ Hence,

$$\mathcal{M}_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (p_a - p_c)^2 - m_\chi^2} = \frac{g_a g_b}{(\mathbf{p}_a - \mathbf{p}_c)^2 - m_\chi^2}$$

Finally, using the expression of the four-momentum of the exchanged particle $\chi$ as $\mathbf{q} = \mathbf{p}_a - \mathbf{p}_c$,

$$\mathcal{M}_{fi} = \frac{g_a g_b}{\mathbf{q}^2 - m_\chi^2}$$

The $g_a g_b$ is related to the interaction vertices, and the term $\frac{1}{\mathbf{q}^2 - m_\chi^2}$ is the propagator.

# Dirac vs Schrödinger

## Schrödinger

The Hamiltonian of the free-particle Schrödinger equation has the form $\hat{H}_S = \frac{\hat{p}^2}{2m}$ and commutes with the angular momentum operator $\hat{L} = \hat{r} \times \hat{p}$. As $\frac{dO}{dt} = i < \psi|[\hat{H}, \hat{O}]|\psi >$, angular momentum is a conserved quantity in non-relativistic quantum mechanics.

## Dirac

The free particle Hamiltonian is $\hat{H}_D = \alpha.\hat{p} + \beta m$. The commutator $[\hat{H}_D, \hat{L}] = [\alpha.\hat{p}, \hat{r} \times \hat{p}]$ can be determined to be

$$[\hat{H}_D, \hat{L}] = -i\alpha \times \hat{p}$$

Hence, the *orbital* angular momentum does *not* commute with the Dirac Hamiltonian and it is *not* a conserved quantity.

The same exercise on the *intrinsic* momentum (spin) $\hat{S} = 1/2\hat{\Sigma} = 1/2 \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$ leads to

$$[\hat{H}_D, \hat{S}] = i\alpha \times \hat{p}$$

Hence, $\hat{S}$ does not commute either and does not correspond to a conserved quantity.

# Dirac vs Schrödinger

## Dirac

But, it is easy to see that

$$[\hat{H}_D, \hat{J}] = 0$$

if

$$\hat{J} = \hat{L} + \hat{S}$$

The *total* angular momentum is conserved.

## Spin of the Dirac particles

Moreover, applying $\hat{S}^2$ on a Dirac Spinor gives $\hat{S}^2 = s(s+1)\psi = 3/4\psi$ The particle solutions of the Dirac equation have an intrinsic angular momentum

$$s = 1/2$$

# Helicity vs chirality

## Why the helicity ?

Let's project the spin on the particle along its direction of propagation and normalize by the momentum

$$h = \frac{S.p}{\mathbf{p}}$$

Taking a four-component Dirac spinor, one gets

$$\hat{h} = \frac{\hat{\Sigma}.\hat{p}}{2\mathbf{p}} = 1/(2\mathbf{p}) \begin{pmatrix} \sigma.\hat{p} & 0 \\ 0 & \sigma.\hat{p} \end{pmatrix}$$

## Helicity and Dirac Hamiltonian eigentstates

From the form of the Dirac Hamiltonian, it can be shown that

$$[\hat{H}_D, \hat{\Sigma}.\hat{p}] = 0$$

Hence, we can find a basis of the spinors which are eigenstates of the Dirac Hamiltonian and of the helicity operators. For spin-half particles, we get left-handed and right-handed helicity states. However, the helicity is not Lorentz invariant and the helicity states should not be confused with the Chirality states (eigenstates of the $\gamma^5$ matrix). The two notions overlap when $E >> m$.

### External lines

| | |
|---|---|
| initial state electron | $u(p)$ |
| initial state positron | $\bar{v}(p)$ |
| initial state photon | $\epsilon^{\mu}$ |
| final state electron | $\bar{u}(p)$ |
| final state positron | $v(p)$ |
| final state photon | $\epsilon^{\mu\star}$ |

### Internal lines and vertex

| | |
|---|---|
| virtual photon | $\frac{-ig_{\mu\nu}}{k^2+i\epsilon}$ |
| virtual electron | $i\frac{\not{p}+m}{p^2-m^2+i\epsilon}$ |
| interaction (vertex) | $ie\gamma^{\mu}$ |

### Matrix element

$$|\mathcal{M}|^2 = \sum_{fi}{}' T_{fi} T_{fi}^{\dagger}$$

Sum over final state, average over initial state

# Moeller Scattering

## Moeller Scattering $e^- e^- \to e^- e^-$

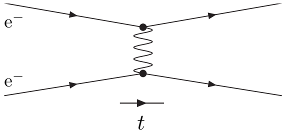
- Simplest diagram with initial and final state of two electrons


$e^-(\mathbf{p_1})e^-(\mathbf{p_2}) \to e^-(\mathbf{p_3})e^-(\mathbf{p_4})$ $e^-(\mathbf{p_1})e^-(\mathbf{p_2}) \to e^-(\mathbf{p_4})e^-(\mathbf{p_3})$

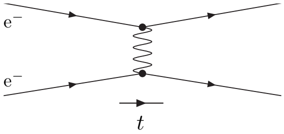
# Moeller Scattering

## Moeller Scattering $e^- e^- \to e^- e^-$

- Simplest diagram with initial and final state of two electrons
- conserve electric charge and momentum at each vertex




$$e^-(\mathbf{p_1})e^-(\mathbf{p_2}) \to e^-(\mathbf{p_3})e^-(\mathbf{p_4})$$

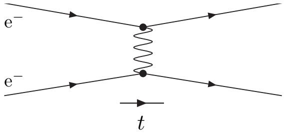



$$e^-(\mathbf{p_1})e^-(\mathbf{p_2}) \to e^-(\mathbf{p_4})e^-(\mathbf{p_3})$$

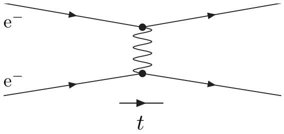
# Moeller Scattering

## Moeller Scattering $e^- e^- \to e^- e^-$

- Simplest diagram with initial and final state of two electrons
- conserve electric charge and momentum at each vertex
- t channel only: $C(e^- + e^-) = -2e \neq C(\gamma) = 0$




$e^-(\mathbf{p_1})e^-(\mathbf{p_2}) \to e^-(\mathbf{p_3})e^-(\mathbf{p_4})$

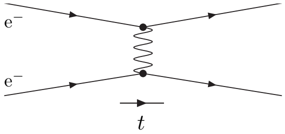



$e^-(\mathbf{p_1})e^-(\mathbf{p_2}) \to e^-(\mathbf{p_4})e^-(\mathbf{p_3})$

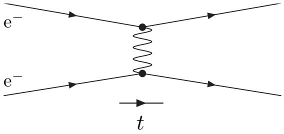
# Moeller Scattering

## Moeller Scattering $e^-e^- \to e^-e^-$

- Simplest diagram with initial and final state of two electrons
- conserve electric charge and momentum at each vertex
- t channel only: $C(e^- + e^-) = -2e \neq C(\gamma) = 0$
- **p** conservation at each vertex $\to$ 2 diagrams $q_\gamma = \mathbf{p_2} - \mathbf{p_3} \neq \mathbf{p_2} - \mathbf{p_4}$

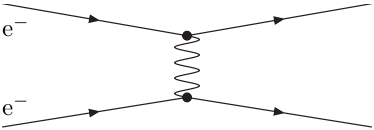



$e^-(\mathbf{p_1})e^-(\mathbf{p_2}) \to e^-(\mathbf{p_3})e^-(\mathbf{p_4})$




$e^-(\mathbf{p_1})e^-(\mathbf{p_2}) \to e^-(\mathbf{p_4})e^-(\mathbf{p_3})$

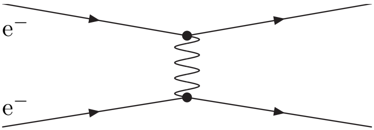
# Moeller Scattering




- Fermion arrow tip to end

$$T_{fi} = [ \qquad ]$$

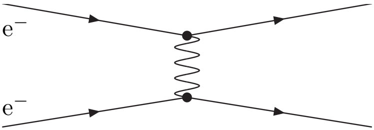
# Moeller Scattering




- Fermion arrow tip to end

$$T_{fi} = [ \quad \bar{u}(\mathbf{p_4}) \qquad \bar{u}(\mathbf{p_3}) \qquad ]$$

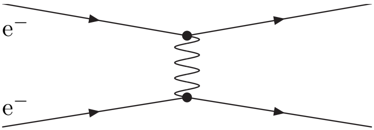
# Moeller Scattering




- Fermion arrow tip to end
- Interaction

$$T_{fi} = [ \quad \bar{u}(\mathbf{p_4}) \qquad \qquad \bar{u}(\mathbf{p_3}) \qquad ]$$

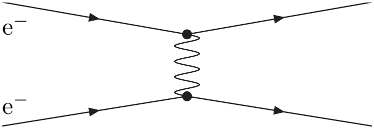
# Moeller Scattering




- Fermion arrow tip to end
- Interaction

$$T_{fi} = [ \quad \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu}) \qquad \bar{u}(\mathbf{p_3})(-ie\gamma^{\nu}) \qquad ]$$

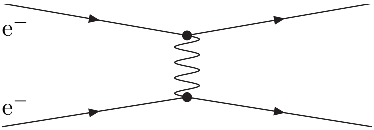
# Moeller Scattering




- Fermion arrow tip to end
- Interaction

$$T_{fi} = [ \quad \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1}) \qquad \bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2}) \quad ]$$

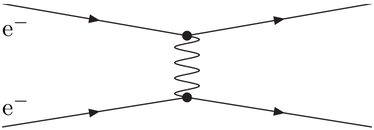
# Moeller Scattering




- Fermion arrow tip to end
- Interaction
- propagator (internal line)

$$T_{fi} \quad = \quad [ \quad \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1}) \qquad \bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2}) \quad ]$$

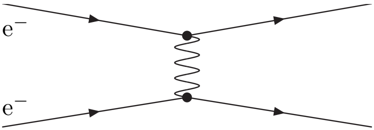
# Moeller Scattering




- Fermion arrow tip to end
- Interaction
- propagator (internal line)

$$T_{fi} = [ \quad \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{k^2})\bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2}) \quad ]$$

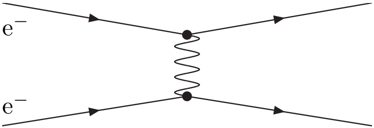
# Moeller Scattering




- Fermion arrow tip to end
- Interaction
- propagator (internal line)
- second graph $\mathbf{p_3} \leftrightarrow \mathbf{p_4}$

$$T_{fi} = [ \quad \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{k^2})\bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2}) \quad ]$$

# Moeller Scattering




- Fermion arrow tip to end
- Interaction
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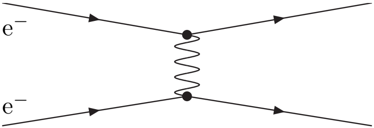
- second graph $\mathbf{p_3} \leftrightarrow \mathbf{p_4}$

$$T_{fi} = [ \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{k^2})\bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2})$$
$$\bar{u}(\mathbf{p_3})(-ie\gamma^{\rho})u(\mathbf{p_1})(\frac{-ig_{\rho\sigma}}{k^2})\bar{u}(\mathbf{p_4})(-ie\gamma^{\sigma})u(\mathbf{p_2})]$$

# Moeller Scattering




- Fermion arrow tip to end
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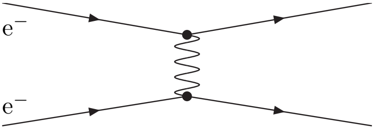
- graphs fermion permutation: $-$

$$T_{fi} = [\quad \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{k^2})\bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2})$$
$$\bar{u}(\mathbf{p_3})(-ie\gamma^{\rho})u(\mathbf{p_1})(\frac{-ig_{\rho\sigma}}{k^2})\bar{u}(\mathbf{p_4})(-ie\gamma^{\sigma})u(\mathbf{p_2})]$$

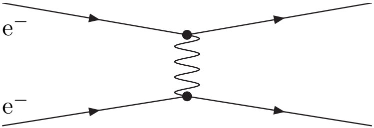
# Moeller Scattering




- Fermion arrow tip to end
- Interaction
- propagator (internal line)
- second graph $\mathbf{p_3} \leftrightarrow \mathbf{p_4}$
- graphs fermion permutation: $-$

$$
\begin{aligned}
T_{fi} = [ \quad & \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{k^2})\bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2}) \\
- \quad & \bar{u}(\mathbf{p_3})(-ie\gamma^{\rho})u(\mathbf{p_1})(\frac{-ig_{\rho\sigma}}{k^2})\bar{u}(\mathbf{p_4})(-ie\gamma^{\sigma})u(\mathbf{p_2})]
\end{aligned}
$$

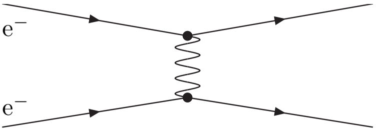
# Moeller Scattering




- Fermion arrow tip to end
- Interaction
- propagator (internal line)
- second graph $\mathbf{p_3} \leftrightarrow \mathbf{p_4}$
- graphs fermion permutation: $-$
- $\mathbf{k} = f(\mathbf{p_i})$

$$T_{fi} = [ \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{k^2})\bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2})$$
$$- \bar{u}(\mathbf{p_3})(-ie\gamma^{\rho})u(\mathbf{p_1})(\frac{-ig_{\rho\sigma}}{k^2})\bar{u}(\mathbf{p_4})(-ie\gamma^{\sigma})u(\mathbf{p_2})]$$

# Moeller Scattering




- Fermion arrow tip to end
- Interaction
- propagator (internal line)
- second graph $\mathbf{p_3} \leftrightarrow \mathbf{p_4}$
- graphs fermion permutation: $-$
- $\mathbf{k} = f(\mathbf{p_i})$

$$
\begin{aligned}
T_{fi} \quad = \quad [ \quad & \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{(\mathbf{p_4}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2}) \\
- \quad & \bar{u}(\mathbf{p_3})(-ie\gamma^{\rho})u(\mathbf{p_1})(\frac{-ig_{\rho\sigma}}{(\mathbf{p_3}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_4})(-ie\gamma^{\sigma})u(\mathbf{p_2})]
\end{aligned}
$$

# Moeller Scattering

$$
\begin{aligned}
\frac{1}{i}T_{fi} &= \frac{1}{i}[ && \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{(\mathbf{p_4}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2}) \\
& - && \bar{u}(\mathbf{p_3})(-ie\gamma^{\rho})u(\mathbf{p_1})(\frac{-ig_{\rho\sigma}}{(\mathbf{p_3}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_4})(-ie\gamma^{\sigma})u(\mathbf{p_2})]
\end{aligned}
$$

# Moeller Scattering

$$
\begin{aligned}
\frac{1}{i}T_{fi} &= \frac{1}{i}[ \quad \bar{u}(\mathbf{p_4})(-ie\gamma^\mu)u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{(\mathbf{p_4}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_3})(-ie\gamma^\nu)u(\mathbf{p_2}) \\
&\quad - \quad \bar{u}(\mathbf{p_3})(-ie\gamma^\rho)u(\mathbf{p_1})(\frac{-ig_{\rho\sigma}}{(\mathbf{p_3}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_4})(-ie\gamma^\sigma)u(\mathbf{p_2})] \\
&= e^2[ \quad \bar{u}(\mathbf{p_4})\gamma^\mu u(\mathbf{p_1})(\frac{g_{\mu\nu}}{(\mathbf{p_4}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_3})\gamma^\nu u(\mathbf{p_2}) \\
&\quad - \quad \bar{u}(\mathbf{p_3})\gamma^\rho u(\mathbf{p_1})(\frac{g_{\rho\sigma}}{(\mathbf{p_3}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_4})\gamma^\sigma u(\mathbf{p_2})]
\end{aligned}
$$

# Moeller Scattering

$$\begin{aligned}
\frac{1}{i}T_{fi} &= \frac{1}{i}[ \quad \bar{u}(\mathbf{p_4})(-ie\gamma^\mu)u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{(\mathbf{p_4}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_3})(-ie\gamma^\nu)u(\mathbf{p_2}) \\
&\quad - \quad \bar{u}(\mathbf{p_3})(-ie\gamma^\rho)u(\mathbf{p_1})(\frac{-ig_{\rho\sigma}}{(\mathbf{p_3}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_4})(-ie\gamma^\sigma)u(\mathbf{p_2})] \\
&= e^2[ \quad \bar{u}(\mathbf{p_4})\gamma^\mu u(\mathbf{p_1})(\frac{g_{\mu\nu}}{(\mathbf{p_4}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_3})\gamma^\nu u(\mathbf{p_2}) \\
&\quad - \quad \bar{u}(\mathbf{p_3})\gamma^\rho u(\mathbf{p_1})(\frac{g_{\rho\sigma}}{(\mathbf{p_3}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_4})\gamma^\sigma u(\mathbf{p_2})] \\
|\mathcal{M}|^2 &= \sum'_{fi} \quad T_{fi}T^\dagger_{fi}
\end{aligned}$$

# Moeller Scattering

$$
\begin{aligned}
\frac{1}{i}T_{fi} &= \frac{1}{i}[ && \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{(\mathbf{p_4}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2}) \\
& - && \bar{u}(\mathbf{p_3})(-ie\gamma^{\rho})u(\mathbf{p_1})(\frac{-ig_{\rho\sigma}}{(\mathbf{p_3}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_4})(-ie\gamma^{\sigma})u(\mathbf{p_2})] \\
&= e^2[ && \bar{u}(\mathbf{p_4})\gamma^{\mu}u(\mathbf{p_1})(\frac{g_{\mu\nu}}{(\mathbf{p_4}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_3})\gamma^{\nu}u(\mathbf{p_2}) \\
& - && \bar{u}(\mathbf{p_3})\gamma^{\rho}u(\mathbf{p_1})(\frac{g_{\rho\sigma}}{(\mathbf{p_3}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_4})\gamma^{\sigma}u(\mathbf{p_2})] \\
|\mathcal{M}|^2 &= \sum\nolimits'_{fi} && T_{fi}T^{\dagger}_{fi} \\
&= \frac{1}{4}\sum\nolimits_{fi} && T_{fi}T^{\dagger}_{fi}
\end{aligned}
$$

# Moeller Scattering

$$
\begin{aligned}
\frac{1}{i}T_{fi} &= \frac{1}{i}[ && \bar{u}(\mathbf{p_4})(-ie\gamma^{\mu})u(\mathbf{p_1})(\frac{-ig_{\mu\nu}}{(\mathbf{p_4}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_3})(-ie\gamma^{\nu})u(\mathbf{p_2}) \\
& - && \bar{u}(\mathbf{p_3})(-ie\gamma^{\rho})u(\mathbf{p_1})(\frac{-ig_{\rho\sigma}}{(\mathbf{p_3}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_4})(-ie\gamma^{\sigma})u(\mathbf{p_2})] \\
&= e^2[ && \bar{u}(\mathbf{p_4})\gamma^{\mu}u(\mathbf{p_1})(\frac{g_{\mu\nu}}{(\mathbf{p_4}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_3})\gamma^{\nu}u(\mathbf{p_2}) \\
& - && \bar{u}(\mathbf{p_3})\gamma^{\rho}u(\mathbf{p_1})(\frac{g_{\rho\sigma}}{(\mathbf{p_3}-\mathbf{p_1})^2})\bar{u}(\mathbf{p_4})\gamma^{\sigma}u(\mathbf{p_2})] \\
|\mathcal{M}|^2 &= \sum'_{fi} && T_{fi}T^{\dagger}_{fi} \\
&= \frac{1}{4}\sum_{fi} && T_{fi}T^{\dagger}_{fi}
\end{aligned}
$$

After a certain number of steps...

$$
|\mathcal{M}|^2 = \frac{64\pi^2\alpha^2}{t^2u^2}[(s-2m^2)^2(t^2+u^2)+ut(-4m^2s+12m^4+ut)]
$$

# Moeller Scattering

$$\frac{d\sigma}{d\Omega} = |\mathcal{M}|^2 \frac{1}{64\pi^2 s}$$

$0 \leq \theta \leq \pi/2$ (electrons)

# Moeller Scattering

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$0 \leq \theta \leq \pi/2$ (electrons) $m_e \approx 0$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{st^2u^2}[s^2(t^2+u^2)+u^2t^2]$$

$$= \frac{\alpha^2}{s}[\frac{s^2}{u^2}+\frac{s^2}{t^2}+1]$$

# Moeller Scattering

$$\frac{d\sigma}{d\Omega} = |\mathcal{M}|^2 \frac{1}{64\pi^2 s}$$

$0 \leq \theta \leq \pi/2$ (electrons) $m_e \approx 0$

$$t = -2\mathbf{p_1 p_3} = -2(\sqrt{s}/2\sqrt{s}/2 - s/4\cos\theta) = -s/2(1-\cos\theta)$$

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{st^2u^2}[s^2(t^2+u^2)+u^2t^2] \\ &= \frac{\alpha^2}{s}[\frac{s^2}{u^2}+\frac{s^2}{t^2}+1] \end{aligned}$$

# Moeller Scattering

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$$t = -2\mathbf{p_1 p_3} = -2(\sqrt{s}/2\sqrt{s}/2 - s/4\cos\theta) = -s/2(1-\cos\theta)$$

$$u = -2\mathbf{p_1 p_4} = -2(s/4 - \vec{p}_1\vec{p}_4) = -2(s/4 + \vec{p}_1\vec{p}_3)$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{st^2u^2}[s^2(t^2+u^2)+u^2t^2]$$

$$= \frac{\alpha^2}{s}[\frac{s^2}{u^2} + \frac{s^2}{t^2} + 1]$$

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$$\frac{d\sigma}{d\Omega} = |\mathcal{M}|^2 \frac{1}{64\pi^2 s}$$

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t &= -2\mathbf{p_1 p_3} = -2(\sqrt{s}/2\sqrt{s}/2 - s/4\cos\theta) = -s/2(1-\cos\theta) \\
u &= -2\mathbf{p_1 p_4} = -2(s/4 - \vec{p}_1\vec{p}_4) = -2(s/4 + \vec{p}_1\vec{p}_3) \\
&= -2(s/4 + s/4\cos\theta) = -s/2(1+\cos\theta) \\
\frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{st^2u^2}[s^2(t^2+u^2) + u^2t^2] \\
&= \frac{\alpha^2}{s}[\frac{s^2}{u^2} + \frac{s^2}{t^2} + 1]
\end{aligned}$$

# Moeller Scattering

$$\frac{d\sigma}{d\Omega} = |\mathcal{M}|^2 \frac{1}{64\pi^2 s}$$

$0 \leq \theta \leq \pi/2$ (electrons) $m_e \approx 0$

$$\begin{aligned}
t &= -2\mathbf{p_1 p_3} = -2(\sqrt{s}/2\sqrt{s}/2 - s/4\cos\theta) = -s/2(1-\cos\theta) \\
u &= -2\mathbf{p_1 p_4} = -2(s/4 - \vec{p}_1\vec{p}_4) = -2(s/4 + \vec{p}_1\vec{p}_3) \\
&= -2(s/4 + s/4\cos\theta) = -s/2(1+\cos\theta) \\
\frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{st^2u^2}[s^2(t^2+u^2) + u^2t^2] \\
&= \frac{\alpha^2}{s}[\frac{s^2}{u^2} + \frac{s^2}{t^2} + 1] \\
&= \frac{\alpha^2}{s}\frac{(3+\cos^2\theta)^2}{\sin^4\theta}
\end{aligned}$$

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$$\frac{d\sigma}{d\Omega} = |\mathcal{M}|^2 \frac{1}{64\pi^2 s}$$

$0 \leq \theta \leq \pi/2$ (electrons) $m_e \approx 0$

$$\begin{aligned}
t &= -2\mathbf{p_1 p_3} = -2(\sqrt{s}/2\sqrt{s}/2 - s/4\cos\theta) = -s/2(1-\cos\theta) \\
u &= -2\mathbf{p_1 p_4} = -2(s/4 - \vec{p}_1\vec{p}_4) = -2(s/4 + \vec{p}_1\vec{p}_3) \\
&= -2(s/4 + s/4\cos\theta) = -s/2(1+\cos\theta) \\
\frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{st^2u^2}[s^2(t^2+u^2) + u^2t^2] \\
&= \frac{\alpha^2}{s}[\frac{s^2}{u^2} + \frac{s^2}{t^2} + 1] \\
&= \frac{\alpha^2}{s}\frac{(3+\cos^2\theta)^2}{\sin^4\theta}
\end{aligned}$$

$s\frac{d\sigma}{d\Omega}$ is scale invariant: measure of the pointlikeness of a particle

# Moeller Scattering

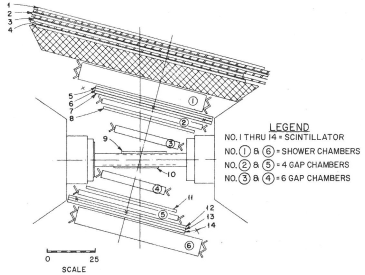



FIG. 1. Storage-ring interaction region and detector system for 556-MeV/electron scattering experiment.

- Stanford-Princeton Storage ring
- $2e^-$ beams $\sqrt{s} = 556 MeV$

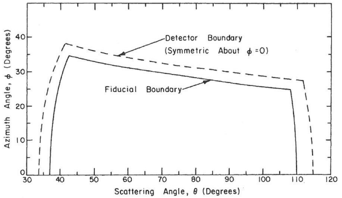



FIG. 2. Detector boundary and fiducial boundaries.

- limited detector acceptance
- differential cross section measurement and prediction

# Moeller Scattering

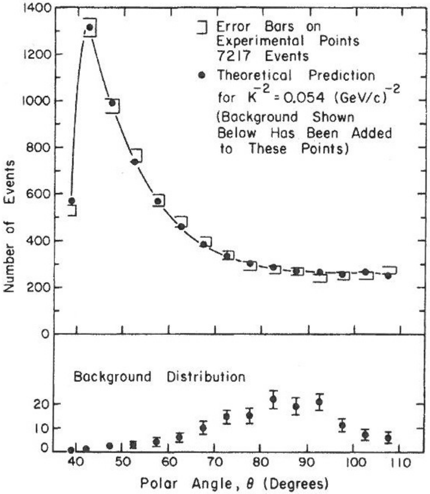


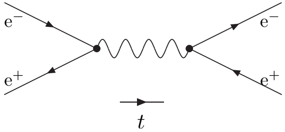

FIG. 3. Comparison of experimental result with Møller scattering modified by radiative corrections. Because the detector geometry is included, the theoretical curve is not symmetric about 90°.

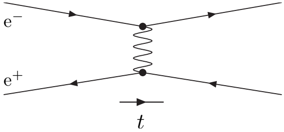
- Typical t channel $\theta = 0 \rightarrow d\sigma/d\Omega \rightarrow \infty$
- Extremely good agreement between the measurement and the theory prediction
- $e^- e^-$ colliders discontinued (1971)

# Bhabha

## The Bhabha Process

Homi Bhabha studied in the 1930s in Great Britain, worked in India afterwards



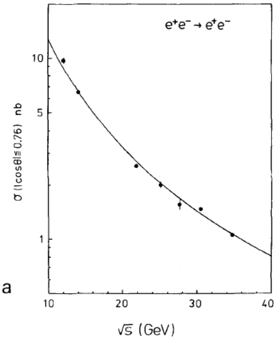

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16s} \frac{(3+\cos^2\theta)^2}{\sin^4\frac{\theta}{2}}$$

- $0 \leq \theta \leq \pi$
- t channel: $\sim \sin^{-4}(\theta/2)$
- s channel: $\sim 1 + \cos^2\theta$

# Bhabha

- PETRA $e^+e^-$ collider $\sqrt{s} \leq 35\mathrm{GeV}$
- JADE, TASSO, CELLO

# Bhabha




- PETRA $\mathrm{e}^{+}\mathrm{e}^{-}$ collider $\sqrt{s} \leq 35\mathrm{GeV}$
- JADE, TASSO, CELLO
- total cross section

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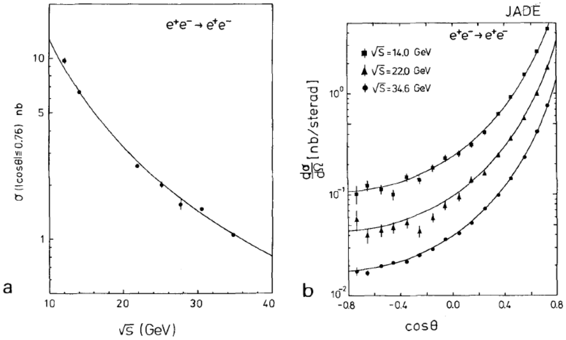
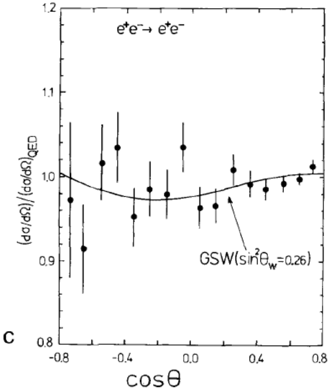



- PETRA $\mathrm{e}^+\mathrm{e}^-$ collider $\sqrt{s} \leq 35\mathrm{GeV}$
- JADE, TASSO, CELLO
- total cross section
- differential cross section

# Bhabha




- Excellent agreement with QED
- Errors reflect statistics
- QED deviation : $s/\Lambda^2 < 5\%$ with $s = 35^2\mathrm{GeV}^2$
- $\rightarrow (\hbar c)/\Lambda = (0.197\mathrm{GeV} \cdot \mathrm{fm})/\Lambda \approx 0.13 \cdot 10^{-3}\mathrm{fm}$
- $N = \int Ldt \cdot \sigma$
- Today Bhabha is a luminosity measurement

# Acceleration

## Electrical field

- acceleration
- charge times potential difference
- typical unit: $eV$

## Magnetic field

- no acceleration
- B field unit: $[B] = \frac{Vs}{m^2}$
- force on charged particle in magnetic field:
  $F = q\vec{v} \times \vec{B} = q\frac{p}{m}B$
- centrifugal force:
  $F = mv^2/r = p^2/(m \cdot r)$
- $R = p/(qBc)$ ($c$ because of natural units)

# Acceleration

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- strong fields difficult to achieve (breakdown)
- accelerate successively

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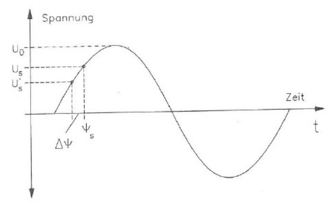
## Phase focussing

- particle sees nominal (not maximal) field
- early particle: less field, less acceleration
- late particle: more field, stronger acceleration



# Acceleration

## LEP/LHC

- circular tunnel 28km circumference
- electron+positron: 210GeV
  - weak field
  - strong cavities
  - energy loss per turn: 6GeV ($\sim E^4/R$)
- LHC proton-proton (13TeV)
  - strong field 10T
  - energy loss per turn: 500keV

## Lepton collider cavities

- LEP: up to 10MV/m
- ILC: 35-40 MV/m
- supraconducting ($T_{He}$)

## Magnetic field LHC

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$$R = \frac{7000 GeV}{0.3 \cdot 10^9 m/s \cdot 10 T \cdot 1 e}$$

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$$= \frac{7000\,GeV}{0.3 \cdot 10^9\, m/s \cdot 10\,Vs/m^2 \cdot 10^{-9}\,Ge}$$

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&= \frac{7000\,GeV}{0.3\cdot 10^9\,m/s\cdot 10\,Vs/m^2\cdot 10^{-9}\,Ge} \\
&\sim 2km \\
R_{true} &\sim 4.5km
\end{aligned}
$$

# Acceleration

## Instantaneous Luminosity

$$L \sim f\frac{n_1 n_2}{\Sigma}$$

$$L \sim \frac{N^2 k_b f \gamma F}{4\pi\epsilon\beta*}$$

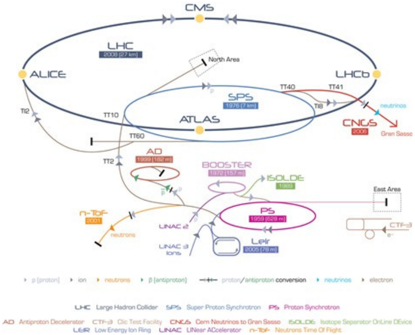
$$\sim \frac{(10^{11})^2 \cdot 2800 \cdot 40MHz \cdot \gamma F}{4\pi \cdot 15\mu m \cdot \beta*}$$

LHC: $10^{34} cm^{-2} s^{-1}$

## Integrated Luminosity

$$N = \int Ldt \cdot \sigma$$

LHC: 25fb$^{-1}$ per experiment



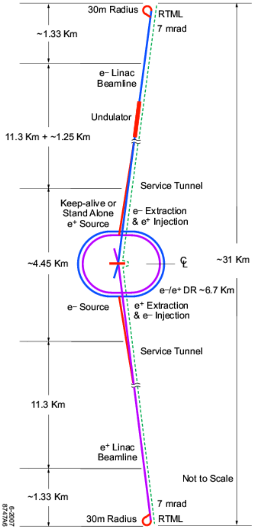

| Linac | Booster | PS | SPS |
|---|---|---|---|
| 50 | 1.4 | 25 | 450 |
| MeV | GeV | GeV | GeV |

6.5 TeV per beam

# Acceleration

## LC the future?

- linear: no synchrotron radiation
- 40km
- polarization
- luminosity
- 250GeV to 1TeV (3TeV: CLIC)



# Detection

## Detection at high energies

- $a + b \rightarrow X \rightarrow \text{neutral} + \text{charged}$

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- particles long-lived wrt detector volume

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- measure points in B-field

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## Electromagnetic calorimeter

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## Electromagnetic calorimeter

- $e + A \rightarrow e + \gamma + A$

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- $\gamma \rightarrow e^+e^-$ etc

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- Tracker: charged particle momenta
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## Tracker

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- highest precision: silicon (dense, $\sim 15\mu m$)
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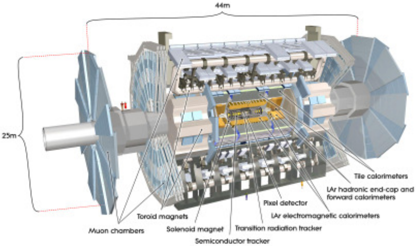
## Electromagnetic calorimeter

- $e + A \rightarrow e + \gamma + A$
- $\gamma \rightarrow e^+e^-$ etc
- shower

# Detection



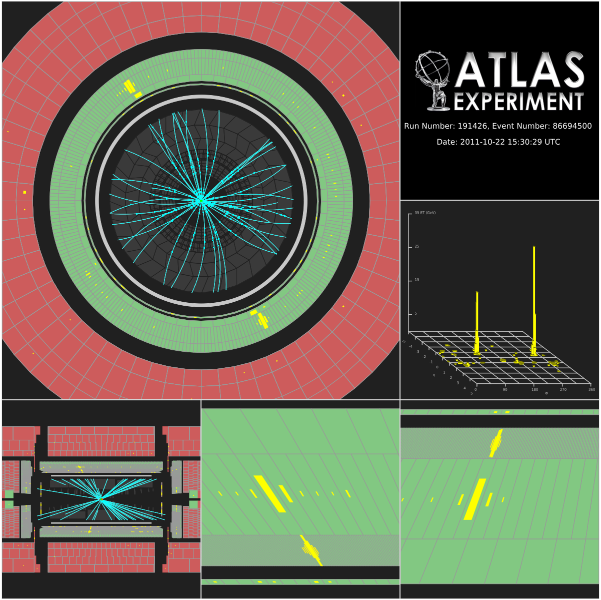

## Experimental Challenges

- bunches every 8m
- 25ns between crossings (fast readout)
- order 20 interactions per crossing
- trigger: 40MHz to 200Hz
- alignment
- calibration

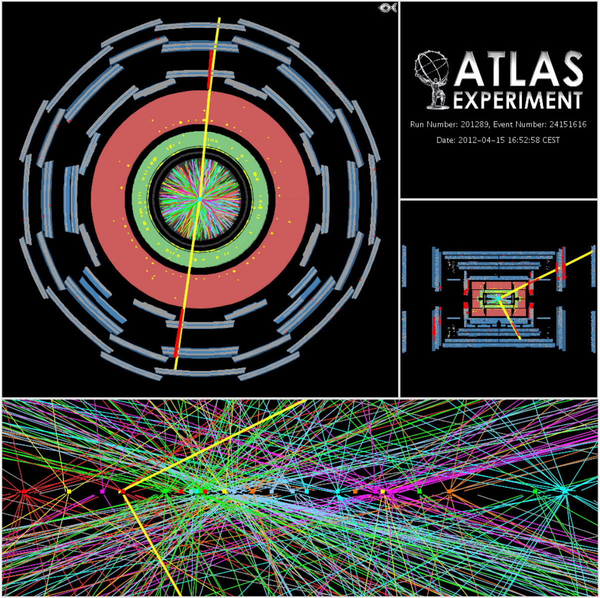
## ATLAS

- Silicon tracking (100M channels 2T)
- Calorimeter (100k)
- Muon chambers (toroid)

# Detection

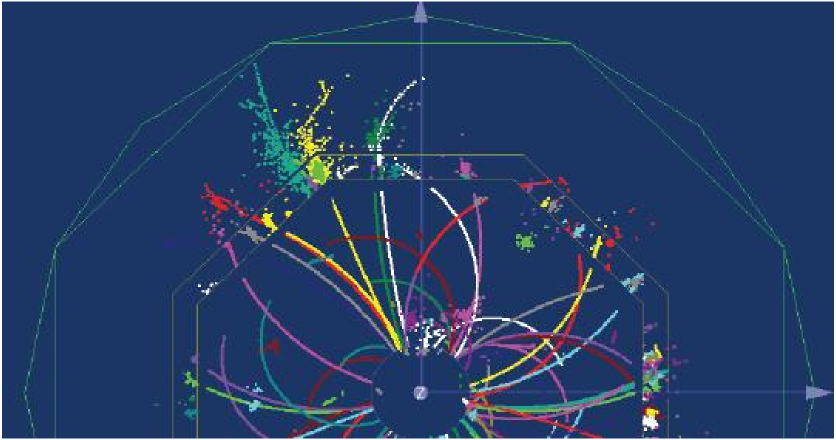

ATLAS EXPERIMENT
Run Number: 191426, Event Number: 86694500
Date: 2011-10-22 15:30:29 UTC

# Detection


ATLAS
EXPERIMENT
Run Number: 201289, Event Number: 24151616
Date: 2012-04-15 16:52:58 CEST

# Detection

## A calorimeter tracker for the future?



# Introduction

## Reminder

- Remember the particle zoo

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \begin{pmatrix} c_L \\ s_L \end{pmatrix} \quad \begin{pmatrix} t_L \\ b_L \end{pmatrix}$$

$$\begin{pmatrix} \nu_{e_L} \\ e_L \end{pmatrix} \quad \begin{pmatrix} \nu_{\mu_L} \\ \mu_L \end{pmatrix} \quad \begin{pmatrix} \nu_{\tau_L} \\ \tau_L \end{pmatrix}$$

$$\begin{matrix} u_R & c_R & t_R \\ d_R & s_R & b_R \\ e_R & \mu_R & \tau_R \end{matrix}$$

$$\begin{matrix} \gamma \\ g \\ W^{\pm}, Z^{\circ} \\ H \end{matrix}$$

# Introduction
## Reminder

- Remember the particle zoo
- $\gamma$ and e

$$\begin{pmatrix} \\ e_L \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix}$$

$$e_R$$

$$\gamma$$

# Introduction

## Reminder

- Remember the particle zoo
- $\gamma$ and $\mathrm{e}$
- today: add $\mu$ and $\tau$

$$\begin{pmatrix} \\ \mathrm{e}_\mathrm{L} \end{pmatrix} \quad \begin{pmatrix} \\ \mu_\mathrm{L} \end{pmatrix} \quad \begin{pmatrix} \\ \tau_\mathrm{L} \end{pmatrix}$$

$$\mathrm{e}_\mathrm{R} \quad \mu_\mathrm{R} \quad \tau_\mathrm{R}$$

$$\gamma$$

# Introduction

## Reminder

- Remember the particle zoo
- $\gamma$ and $\mathrm{e}$
- today: add $\mu$ and $\tau$

**Definition**

Charged Leptons: $\mathrm{e}$, $\mu$, $\tau$
Leptons: charged leptons plus neutrinos
Jargon: leptons as charged leptons

$$\begin{pmatrix} \\ \mathrm{e_L} \end{pmatrix} \quad \begin{pmatrix} \\ \mu_\mathrm{L} \end{pmatrix} \quad \begin{pmatrix} \\ \tau_\mathrm{L} \end{pmatrix}$$

$$\mathrm{e_R} \quad \mu_\mathrm{R} \quad \tau_\mathrm{R}$$

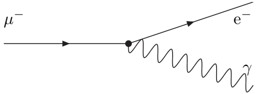
$$\gamma$$

# Introduction

## Muon properties

### Properties of the $\mu$

$$
\begin{aligned}
m_0 &= 0.105\mathrm{GeV} && \mu^+ e^- \\
\tau &= (2.197 \cdot 10^{-6})\mathrm{s} && \mathrm{PSI} \\
c\tau &= 659\mathrm{m}
\end{aligned}
$$



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$$
\begin{array}{lll l}
m_0 & = & 1.777\mathrm{GeV} & e^+e^- \\
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c\tau & = & 87\mu\mathrm{m} &
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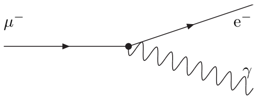
$$



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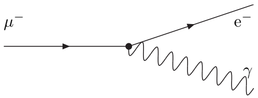


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\begin{array}{lll}
\mathcal{B}(\mu \to e\gamma) & < & 5.7 \cdot 10^{-13} \\
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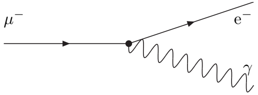


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### Lepton numbers (additive QNs)

| $L_e$ | $L_\mu$ | $L_\tau$ |
|---|---|---|

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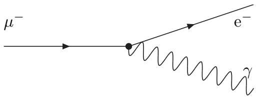


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| $e^-$ | 1 | 0 | 0 |

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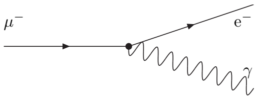


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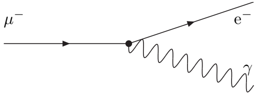
$$\begin{array}{lll l} m_0 & = & 0.105\mathrm{GeV} & \mu^+e^- \\ \tau & = & (2.197\cdot 10^{-6})\mathrm{s} & \mathrm{PSI} \\ c\tau & = & 659\mathrm{m} & \end{array}$$

### Properties of the $\tau$

$$\begin{array}{lll l} m_0 & = & 1.777\mathrm{GeV} & e^+e^- \\ \tau & = & (2.906\cdot 10^{-13})\mathrm{s} & e^+e^- \\ c\tau & = & 87\mu\mathrm{m} & \end{array}$$




$$\begin{array}{lll} \mathcal{B}(\mu\to e\gamma) & < & 5.7\cdot 10^{-13} \\ \mathcal{B}(\tau\to e\gamma) & < & 3.3\cdot 10^{-8} \\ \mathcal{B}(\tau\to \mu\gamma) & < & 4.4\cdot 10^{-8} \\ CL & = & 90\% \end{array}$$

### Lepton numbers (additive QNs)

| | $L_e$ | $L_\mu$ | $L_\tau$ |
|---|---|---|---|
| $e^-$ | 1 | 0 | 0 |
| $e^+$ | −1 | 0 | 0 |
| $\mu^-$ | 0 | 1 | 0 |
| $\mu^+$ | 0 | −1 | 0 |
| $\tau^-$ | 0 | 0 | 1 |
| $\tau^+$ | 0 | 0 | −1 |

# Introduction

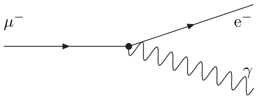
## Muon properties

### Properties of the $\mu$

$$\begin{aligned} m_0 &= 0.105\text{GeV} & \mu^+e^- \\ \tau &= (2.197 \cdot 10^{-6})\text{s} & \text{PSI} \\ c\tau &= 659\text{m} \end{aligned}$$

### Properties of the $\tau$

$$\begin{aligned} m_0 &= 1.777\text{GeV} & e^+e^- \\ \tau &= (2.906 \cdot 10^{-13})\text{s} & e^+e^- \\ c\tau &= 87\mu\text{m} \end{aligned}$$

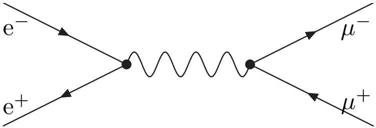



$$\begin{aligned} \mathcal{B}(\mu \to e\gamma) &< 5.7 \cdot 10^{-13} \\ \mathcal{B}(\tau \to e\gamma) &< 3.3 \cdot 10^{-8} \\ \mathcal{B}(\tau \to \mu\gamma) &< 4.4 \cdot 10^{-8} \\ CL &= 90\% \end{aligned}$$

### Lepton numbers (additive QNs)

| | $L_e$ | $L_\mu$ | $L_\tau$ |
|---|---|---|---|
| $e^-$ | 1 | 0 | 0 |
| $e^+$ | $-1$ | 0 | 0 |
| $\mu^-$ | 0 | 1 | 0 |
| $\mu^+$ | 0 | $-1$ | 0 |
| $\tau^-$ | 0 | 0 | 1 |
| $\tau^+$ | 0 | 0 | $-1$ |
| non $-$ leptons | 0 | 0 | 0 |

# Process calculation

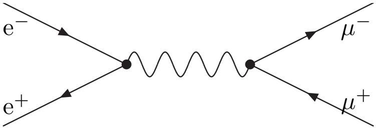



$$e^+(\mathbf{p_2})e^-(\mathbf{p_1}) \rightarrow \mu^+(\mathbf{p_4})\mu^-(\mathbf{p_3})$$

## Transition Amplitude

[ ]

# Process calculation



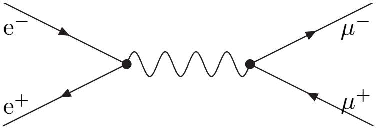

- $L_e^i = 1 - 1 = 0 = L_e^f$

$$e^+(\mathbf{p_2})e^-(\mathbf{p_1}) \rightarrow \mu^+(\mathbf{p_4})\mu^-(\mathbf{p_3})$$

## Transition Amplitude

$$[ \qquad\qquad ]$$

# Process calculation



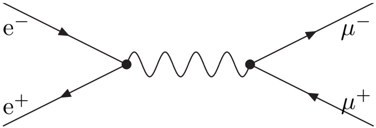

- $L_e^i = 1 - 1 = 0 = L_e^f$
- $L_\mu^i = 0 = 1 - 1 = L_\mu^f$

$$e^+(\mathbf{p_2})e^-(\mathbf{p_1}) \to \mu^+(\mathbf{p_4})\mu^-(\mathbf{p_3})$$

## Transition Amplitude

[ ]

# Process calculation



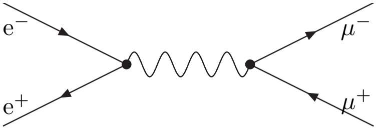

- $L_e^i = 1 - 1 = 0 = L_e^f$
- $L_\mu^i = 0 = 1 - 1 = L_\mu^f$
- Initial state

$$e^+(\mathbf{p_2})e^-(\mathbf{p_1}) \to \mu^+(\mathbf{p_4})\mu^-(\mathbf{p_3})$$

## Transition Amplitude

[ ]

# Process calculation



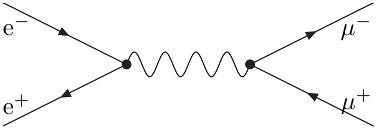

- $L_e^i = 1 - 1 = 0 = L_e^f$
- $L_\mu^i = 0 = 1 - 1 = L_\mu^f$
- Initial state

$$e^+(\mathbf{p_2})e^-(\mathbf{p_1}) \to \mu^+(\mathbf{p_4})\mu^-(\mathbf{p_3})$$

## Transition Amplitude

$$\frac{1}{i}T_{fi} = \frac{1}{i}[\bar{v}(\mathbf{p_2})(-ie\gamma^\mu)u(\mathbf{p_1}) \qquad ]$$

# Process calculation



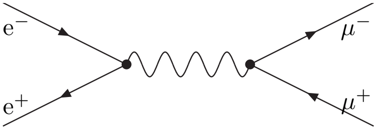

- $L_e^i = 1 - 1 = 0 = L_e^f$
- $L_\mu^i = 0 = 1 - 1 = L_\mu^f$
- Initial state
- Final state

$$e^+(\mathbf{p_2})e^-(\mathbf{p_1}) \rightarrow \mu^+(\mathbf{p_4})\mu^-(\mathbf{p_3})$$

## Transition Amplitude

$$\frac{1}{i}T_{fi} \quad = \quad \frac{1}{i}[\bar{v}(\mathbf{p_2})(-ie\gamma^\mu)u(\mathbf{p_1}) \qquad\qquad ]$$

# Process calculation



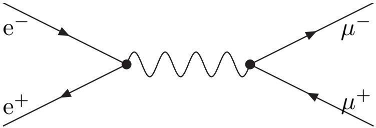

- $L_e^i = 1 - 1 = 0 = L_e^f$
- $L_\mu^i = 0 = 1 - 1 = L_\mu^f$
- Initial state
- Final state

$$e^+(\mathbf{p_2})e^-(\mathbf{p_1}) \rightarrow \mu^+(\mathbf{p_4})\mu^-(\mathbf{p_3})$$

## Transition Amplitude

$$\frac{1}{i}T_{fi} = \frac{1}{i}[\bar{v}(\mathbf{p_2})(-ie\gamma^\mu)u(\mathbf{p_1}) \qquad \bar{u}(\mathbf{p_3})(-ie\gamma^\nu)v(\mathbf{p_4})]$$

# Process calculation



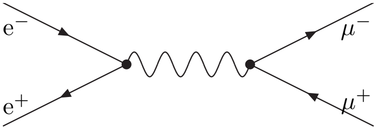

- $L_e^i = 1 - 1 = 0 = L_e^f$
- $L_\mu^i = 0 = 1 - 1 = L_\mu^f$
- Initial state
- Final state
- Photon Propagator

$$e^+(\mathbf{p_2})e^-(\mathbf{p_1}) \rightarrow \mu^+(\mathbf{p_4})\mu^-(\mathbf{p_3})$$

## Transition Amplitude

$$\frac{1}{i}T_{fi} = \frac{1}{i}[\bar{v}(\mathbf{p_2})(-ie\gamma^\mu)u(\mathbf{p_1}) \qquad \bar{u}(\mathbf{p_3})(-ie\gamma^\nu)v(\mathbf{p_4})]$$

# Process calculation



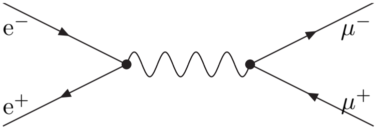

- $L_e^i = 1 - 1 = 0 = L_e^f$
- $L_\mu^i = 0 = 1 - 1 = L_\mu^f$
- Initial state
- Final state
- Photon Propagator

$$e^+(\mathbf{p_2})e^-(\mathbf{p_1}) \to \mu^+(\mathbf{p_4})\mu^-(\mathbf{p_3})$$

## Transition Amplitude

$$\frac{1}{i}T_{fi} = \frac{1}{i}[\bar{v}(\mathbf{p_2})(-ie\gamma^\mu)u(\mathbf{p_1})\frac{-ig_{\mu\nu}}{k^2}\bar{u}(\mathbf{p_3})(-ie\gamma^\nu)v(\mathbf{p_4})]$$

# Process calculation




- $L_e^i = 1 - 1 = 0 = L_e^f$
- $L_\mu^i = 0 = 1 - 1 = L_\mu^f$
- Initial state
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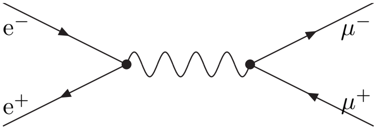
$$e^+(\mathbf{p_2})e^-(\mathbf{p_1}) \to \mu^+(\mathbf{p_4})\mu^-(\mathbf{p_3})$$

## Transition Amplitude

$$\frac{1}{i}T_{fi} = \frac{1}{i}[\bar{v}(\mathbf{p_2})(-ie\gamma^\mu)u(\mathbf{p_1})\frac{-ig_{\mu\nu}}{k^2}\bar{u}(\mathbf{p_3})(-ie\gamma^\nu)v(\mathbf{p_4})]$$

$$= e^2[\bar{v}(\mathbf{p_2})\gamma^\mu u(\mathbf{p_1})\frac{g_{\mu\nu}}{s}\bar{u}(\mathbf{p_3})\gamma^\nu v(\mathbf{p_4})]$$

# Process calculation




- $L_e^i = 1 - 1 = 0 = L_e^f$
- $L_\mu^i = 0 = 1 - 1 = L_\mu^f$
- Initial state
- Final state
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$$e^+(\mathbf{p_2})e^-(\mathbf{p_1}) \rightarrow \mu^+(\mathbf{p_4})\mu^-(\mathbf{p_3})$$

## Transition Amplitude

$$\begin{aligned}
\frac{1}{i} T_{fi} &= \frac{1}{i}[\bar{v}(\mathbf{p_2})(-ie\gamma^\mu)u(\mathbf{p_1})\frac{-ig_{\mu\nu}}{k^2}\bar{u}(\mathbf{p_3})(-ie\gamma^\nu)v(\mathbf{p_4})] \\
&= e^2[\bar{v}(\mathbf{p_2})\gamma^\mu u(\mathbf{p_1})\frac{g_{\mu\nu}}{s}\bar{u}(\mathbf{p_3})\gamma^\nu v(\mathbf{p_4})] \\
&= \frac{e^2}{s}[\bar{v}(\mathbf{p_2})\gamma^\mu u(\mathbf{p_1})\bar{u}(\mathbf{p_3})\gamma_\mu v(\mathbf{p_4})]
\end{aligned}$$

# Process calculation

## useful formula

$$
\begin{aligned}
\gamma_0 &= g_{\mu 0}\gamma^0 &&= \gamma^0 \\
\gamma_k &= g_{\mu k}\gamma^k &&= -\gamma^k \\
\bar{u} &= u^\dagger\gamma^0 &&= u^\dagger\gamma_0 \\
(\gamma^\mu)^\dagger &= \gamma^0\gamma^\mu\gamma^0 && \\
(\gamma_\mu)^\dagger &= g_{\mu\nu}(\gamma^\nu)^\dagger = g_{\mu\nu}(\gamma^0\gamma^\nu\gamma^0) &&= \gamma^0\gamma_\mu\gamma^0 \\
\gamma^0\gamma^0 &= 1_4 &&
\end{aligned}
$$

## Element matrix squared

$$[\bar{v}(\mathbf{p_2})\gamma^\nu u(\mathbf{p_1})\bar{u}(\mathbf{p_3})\gamma_\nu v(\mathbf{p_4})]^\dagger$$

# Process calculation

## Useful Formula

$$\begin{aligned}\gamma_0 &= g_{\mu 0}\gamma^0 &= \gamma^0\\ \gamma_k &= g_{\mu k}\gamma^k &= -\gamma^k\\ \bar{u} &= u^\dagger\gamma^0 &= u^\dagger\gamma_0\end{aligned}$$

## Insert

$$[\bar{v}(\mathbf{p_2})\gamma^\nu u(\mathbf{p_1})\bar{u}(\mathbf{p_3})\gamma_\nu v(\mathbf{p_4})]^\dagger$$

# Process calculation

## Useful Formula

## Insert

$$
\begin{aligned}
& [\bar{v}(\mathbf{p_2})\gamma^{\nu}u(\mathbf{p_1})\bar{u}(\mathbf{p_3})\gamma_{\nu}v(\mathbf{p_4})]^{\dagger} \\
= \quad & [v^{\dagger}(\mathbf{p_2})\gamma^{0}\gamma^{\nu}u(\mathbf{p_1})u^{\dagger}(\mathbf{p_3})\gamma^{0}\gamma_{\nu}v(\mathbf{p_4})]^{\dagger}
\end{aligned}
$$

# Process calculation

## Useful Formula

## Insert

$$
\begin{aligned}
& [\bar{v}(\mathbf{p_2})\gamma^{\nu} u(\mathbf{p_1})\bar{u}(\mathbf{p_3})\gamma_{\nu} v(\mathbf{p_4})]^{\dagger} \\
= \; & [v^{\dagger}(\mathbf{p_2})\gamma^{0}\gamma^{\nu} u(\mathbf{p_1})u^{\dagger}(\mathbf{p_3})\gamma^{0}\gamma_{\nu} v(\mathbf{p_4})]^{\dagger} \\
= \; & [v^{\dagger}(\mathbf{p_4})(\gamma_{\nu})^{\dagger}(\gamma^{0})^{\dagger}(u^{\dagger})^{\dagger}(\mathbf{p_3})u^{\dagger}(\mathbf{p_1})(\gamma^{\nu})^{\dagger}(\gamma^{0})^{\dagger}(v^{\dagger})^{\dagger}(\mathbf{p_2})]
\end{aligned}
$$

# Process calculation

## Useful Formula

## Insert

$$
\begin{aligned}
& [\bar{v}(\mathbf{p_2})\gamma^\nu u(\mathbf{p_1})\bar{u}(\mathbf{p_3})\gamma_\nu v(\mathbf{p_4})]^\dagger \\
= \; & [v^\dagger(\mathbf{p_2})\gamma^0\gamma^\nu u(\mathbf{p_1})u^\dagger(\mathbf{p_3})\gamma^0\gamma_\nu v(\mathbf{p_4})]^\dagger \\
= \; & [v^\dagger(\mathbf{p_4})(\gamma_\nu)^\dagger(\gamma^0)^\dagger(u^\dagger)^\dagger(\mathbf{p_3})u^\dagger(\mathbf{p_1})(\gamma^\nu)^\dagger(\gamma^0)^\dagger(v^\dagger)^\dagger(\mathbf{p_2})] \\
= \; & [v^\dagger(\mathbf{p_4})(\gamma_\nu)^\dagger(\gamma^0)^\dagger u(\mathbf{p_3})u^\dagger(\mathbf{p_1})(\gamma^\nu)^\dagger(\gamma^0)^\dagger v(\mathbf{p_2})]
\end{aligned}
$$

# Process calculation

## Useful Formula

$$
\begin{aligned}
(\gamma^\mu)^\dagger &= \gamma^0\gamma^\mu\gamma^0 \\
(\gamma_\mu)^\dagger &= g_{\mu\nu}(\gamma^\nu)^\dagger = g_{\mu\nu}(\gamma^0\gamma^\nu\gamma^0) = \gamma^0\gamma_\mu\gamma^0
\end{aligned}
$$

## Insert

$$
\begin{aligned}
& [\bar{v}(\mathbf{p_2})\gamma^\nu u(\mathbf{p_1})\bar{u}(\mathbf{p_3})\gamma_\nu v(\mathbf{p_4})]^\dagger \\
= & [v^\dagger(\mathbf{p_2})\gamma^0\gamma^\nu u(\mathbf{p_1})u^\dagger(\mathbf{p_3})\gamma^0\gamma_\nu v(\mathbf{p_4})]^\dagger \\
= & [v^\dagger(\mathbf{p_4})(\gamma_\nu)^\dagger(\gamma^0)^\dagger(u^\dagger)^\dagger(\mathbf{p_3})u^\dagger(\mathbf{p_1})(\gamma^\nu)^\dagger(\gamma^0)^\dagger(v^\dagger)^\dagger(\mathbf{p_2})] \\
= & [v^\dagger(\mathbf{p_4})(\gamma_\nu)^\dagger(\gamma^0)^\dagger u(\mathbf{p_3})u^\dagger(\mathbf{p_1})(\gamma^\nu)^\dagger(\gamma^0)^\dagger v(\mathbf{p_2})]
\end{aligned}
$$

# Process calculation

## Useful Formula

$$
\begin{aligned}
(\gamma^\mu)^\dagger &= \gamma^0\gamma^\mu\gamma^0 \\
(\gamma_\mu)^\dagger &= g_{\mu\nu}(\gamma^\nu)^\dagger = g_{\mu\nu}(\gamma^0\gamma^\nu\gamma^0) = \gamma^0\gamma_\mu\gamma^0
\end{aligned}
$$

## Insert

$$
\begin{aligned}
& [\bar{v}(\mathbf{p_2})\gamma^\nu u(\mathbf{p_1})\bar{u}(\mathbf{p_3})\gamma_\nu v(\mathbf{p_4})]^\dagger \\
= & [v^\dagger(\mathbf{p_2})\gamma^0\gamma^\nu u(\mathbf{p_1})u^\dagger(\mathbf{p_3})\gamma^0\gamma_\nu v(\mathbf{p_4})]^\dagger \\
= & [v^\dagger(\mathbf{p_4})(\gamma_\nu)^\dagger(\gamma^0)^\dagger(u^\dagger)^\dagger(\mathbf{p_3})u^\dagger(\mathbf{p_1})(\gamma^\nu)^\dagger(\gamma^0)^\dagger(v^\dagger)^\dagger(\mathbf{p_2})] \\
= & [v^\dagger(\mathbf{p_4})(\gamma_\nu)^\dagger(\gamma^0)^\dagger u(\mathbf{p_3})u^\dagger(\mathbf{p_1})(\gamma^\nu)^\dagger(\gamma^0)^\dagger v(\mathbf{p_2})] \\
= & [v^\dagger(\mathbf{p_4})\gamma^0\gamma_\nu\gamma^0\gamma^0\gamma^0\gamma^0 u(\mathbf{p_3})u^\dagger(\mathbf{p_1})\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^0\gamma^0 v(\mathbf{p_2})]
\end{aligned}
$$

# Process calculation

## Useful Formula

$$\gamma^0\gamma^0 = 1_4$$

## Insert

$$\begin{aligned}
& [\bar{v}(\mathbf{p_2})\gamma^\nu u(\mathbf{p_1})\bar{u}(\mathbf{p_3})\gamma_\nu v(\mathbf{p_4})]^\dagger \\
= & [v^\dagger(\mathbf{p_2})\gamma^0\gamma^\nu u(\mathbf{p_1})u^\dagger(\mathbf{p_3})\gamma^0\gamma_\nu v(\mathbf{p_4})]^\dagger \\
= & [v^\dagger(\mathbf{p_4})(\gamma_\nu)^\dagger(\gamma^0)^\dagger(u^\dagger)^\dagger(\mathbf{p_3})u^\dagger(\mathbf{p_1})(\gamma^\nu)^\dagger(\gamma^0)^\dagger(v^\dagger)^\dagger(\mathbf{p_2})] \\
= & [v^\dagger(\mathbf{p_4})(\gamma_\nu)^\dagger(\gamma^0)^\dagger u(\mathbf{p_3})u^\dagger(\mathbf{p_1})(\gamma^\nu)^\dagger(\gamma^0)^\dagger v(\mathbf{p_2})] \\
= & [v^\dagger(\mathbf{p_4})\gamma^0\gamma_\nu\gamma^0\gamma^0\gamma^0\gamma^0 u(\mathbf{p_3})u^\dagger(\mathbf{p_1})\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^0\gamma^0 v(\mathbf{p_2})]
\end{aligned}$$

# Process calculation

## Useful Formula

$$\gamma^0\gamma^0 = 1_4$$

## Insert

$$\begin{aligned}
& [\bar{v}(\mathbf{p_2})\gamma^\nu u(\mathbf{p_1})\bar{u}(\mathbf{p_3})\gamma_\nu v(\mathbf{p_4})]^\dagger \\
= \; & [v^\dagger(\mathbf{p_2})\gamma^0\gamma^\nu u(\mathbf{p_1})u^\dagger(\mathbf{p_3})\gamma^0\gamma_\nu v(\mathbf{p_4})]^\dagger \\
= \; & [v^\dagger(\mathbf{p_4})(\gamma_\nu)^\dagger(\gamma^0)^\dagger(u^\dagger)^\dagger(\mathbf{p_3})u^\dagger(\mathbf{p_1})(\gamma^\nu)^\dagger(\gamma^0)^\dagger(v^\dagger)^\dagger(\mathbf{p_2})] \\
= \; & [v^\dagger(\mathbf{p_4})(\gamma_\nu)^\dagger(\gamma^0)^\dagger u(\mathbf{p_3})u^\dagger(\mathbf{p_1})(\gamma^\nu)^\dagger(\gamma^0)^\dagger v(\mathbf{p_2})] \\
= \; & [v^\dagger(\mathbf{p_4})\gamma^0\gamma_\nu\gamma^0\gamma^0\gamma^0\gamma^0 u(\mathbf{p_3})u^\dagger(\mathbf{p_1})\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^0\gamma^0 v(\mathbf{p_2})] \\
= \; & [v^\dagger(\mathbf{p_4})\gamma^0\gamma_\nu u(\mathbf{p_3})u^\dagger(\mathbf{p_1})\gamma^0\gamma^\nu v(\mathbf{p_2})]
\end{aligned}$$

# Process calculation

## Useful Formula

## Insert

$$
\begin{aligned}
& [\bar{v}(\mathbf{p_2})\gamma^{\nu}u(\mathbf{p_1})\bar{u}(\mathbf{p_3})\gamma_{\nu}v(\mathbf{p_4})]^{\dagger} \\
= \; & [v^{\dagger}(\mathbf{p_2})\gamma^0\gamma^{\nu}u(\mathbf{p_1})u^{\dagger}(\mathbf{p_3})\gamma^0\gamma_{\nu}v(\mathbf{p_4})]^{\dagger} \\
= \; & [v^{\dagger}(\mathbf{p_4})(\gamma_{\nu})^{\dagger}(\gamma^0)^{\dagger}(u^{\dagger})^{\dagger}(\mathbf{p_3})u^{\dagger}(\mathbf{p_1})(\gamma^{\nu})^{\dagger}(\gamma^0)^{\dagger}(v^{\dagger})^{\dagger}(\mathbf{p_2})] \\
= \; & [v^{\dagger}(\mathbf{p_4})(\gamma_{\nu})^{\dagger}(\gamma^0)^{\dagger}u(\mathbf{p_3})u^{\dagger}(\mathbf{p_1})(\gamma^{\nu})^{\dagger}(\gamma^0)^{\dagger}v(\mathbf{p_2})] \\
= \; & [v^{\dagger}(\mathbf{p_4})\gamma^0\gamma_{\nu}\gamma^0\gamma^0\gamma^0\gamma^0u(\mathbf{p_3})u^{\dagger}(\mathbf{p_1})\gamma^0\gamma^{\nu}\gamma^0\gamma^0\gamma^0\gamma^0v(\mathbf{p_2})] \\
= \; & [v^{\dagger}(\mathbf{p_4})\gamma^0\gamma_{\nu}u(\mathbf{p_3})u^{\dagger}(\mathbf{p_1})\gamma^0\gamma^{\nu}v(\mathbf{p_2})] \\
= \; & [\bar{v}(\mathbf{p_4})\gamma_{\nu}u(\mathbf{p_3})\bar{u}(\mathbf{p_1})\gamma^{\nu}v(\mathbf{p_2})]
\end{aligned}
$$

# Process calculation

## Formula

## Matrix Element

$$|\mathcal{M}|^2 = \frac{e^4}{4s^2}\sum[\bar{v}_a(\mathbf{p_2})\gamma^{\mu}_{ab}u_b(\mathbf{p_1})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd}v_d(\mathbf{p_4})]$$
$$[\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef}u_f(\mathbf{p_3})\bar{u}_g(\mathbf{p_1})\gamma^{\nu}_{gh}v_h(\mathbf{p_2})]$$

# Process calculation

## Formula

## Matrix Element

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{e^4}{4s^2}\sum[\bar{v}_a(\mathbf{p_2})\gamma^{\mu}_{ab}u_b(\mathbf{p_1})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd}v_d(\mathbf{p_4})] \\
&\quad [\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef}u_f(\mathbf{p_3})\bar{u}_g(\mathbf{p_1})\gamma^{\nu}_{gh}v_h(\mathbf{p_2})] \\
&= \frac{e^4}{4s^2}\sum v_h(\mathbf{p_2})\bar{v}_a(\mathbf{p_2})\gamma^{\mu}_{ab}u_b(\mathbf{p_1})\bar{u}_g(\mathbf{p_1})\gamma^{\nu}_{gh} \\
&\quad u_f(\mathbf{p_3})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd}v_d(\mathbf{p_4})\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef}
\end{aligned}
$$

# Process calculation

## Formula

$$\sum_f M_{ff} = Tr(M)$$

## Matrix Element

$$|\mathcal{M}|^2 = \frac{e^4}{4s^2}\sum[\bar{v}_a(\mathbf{p_2})\gamma^{\mu}_{ab}u_b(\mathbf{p_1})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd}v_d(\mathbf{p_4})]$$
$$[\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef}u_f(\mathbf{p_3})\bar{u}_g(\mathbf{p_1})\gamma^{\nu}_{gh}v_h(\mathbf{p_2})]$$
$$= \frac{e^4}{4s^2}\sum v_h(\mathbf{p_2})\bar{v}_a(\mathbf{p_2})\gamma^{\mu}_{ab}u_b(\mathbf{p_1})\bar{u}_g(\mathbf{p_1})\gamma^{\nu}_{gh}$$
$$u_f(\mathbf{p_3})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd}v_d(\mathbf{p_4})\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef}$$

# Process calculation

## Formula

$$\sum_f M_{ff} = Tr(M)$$

## Matrix Element

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{e^4}{4s^2}\sum[\bar{v}_a(\mathbf{p_2})\gamma^{\mu}_{ab}u_b(\mathbf{p_1})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd}v_d(\mathbf{p_4})] \\
&[\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef}u_f(\mathbf{p_3})\bar{u}_g(\mathbf{p_1})\gamma^{\nu}_{gh}v_h(\mathbf{p_2})] \\
&= \frac{e^4}{4s^2}\sum v_h(\mathbf{p_2})\bar{v}_a(\mathbf{p_2})\gamma^{\mu}_{ab}u_b(\mathbf{p_1})\bar{u}_g(\mathbf{p_1})\gamma^{\nu}_{gh} \\
&u_f(\mathbf{p_3})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd}v_d(\mathbf{p_4})\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef} \\
&= \frac{e^4}{4s^2}\sum_s Tr(v(\mathbf{p_2})\bar{v}(\mathbf{p_2})\gamma^{\mu}u(\mathbf{p_1})\bar{u}(\mathbf{p_1})\gamma^{\nu}) \\
&Tr(u(\mathbf{p_3})\bar{u}(\mathbf{p_3})\gamma_{\mu}v(\mathbf{p_4})\bar{v}(\mathbf{p_4})\gamma_{\nu})
\end{aligned}
$$

# Process calculation

## Formula

$$\sum_{spin} u\bar{u} \quad = \quad \sum_{spin} v\bar{v} = \not{p} = p_\nu \gamma^\nu$$

## Matrix Element

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{e^4}{4s^2} \sum [\bar{v}_a(\mathbf{p_2})\gamma^\mu_{ab} u_b(\mathbf{p_1})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd} v_d(\mathbf{p_4})] \\
&\quad [\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef} u_f(\mathbf{p_3})\bar{u}_g(\mathbf{p_1})\gamma^\nu_{gh} v_h(\mathbf{p_2})] \\
&= \frac{e^4}{4s^2} \sum v_h(\mathbf{p_2})\bar{v}_a(\mathbf{p_2})\gamma^\mu_{ab} u_b(\mathbf{p_1})\bar{u}_g(\mathbf{p_1})\gamma^\nu_{gh} \\
&\quad u_f(\mathbf{p_3})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd} v_d(\mathbf{p_4})\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef} \\
&= \frac{e^4}{4s^2} \sum_s Tr(v(\mathbf{p_2})\bar{v}(\mathbf{p_2})\gamma^\mu u(\mathbf{p_1})\bar{u}(\mathbf{p_1})\gamma^\nu) \\
&\quad Tr(u(\mathbf{p_3})\bar{u}(\mathbf{p_3})\gamma_\mu v(\mathbf{p_4})\bar{v}(\mathbf{p_4})\gamma_\nu)
\end{aligned}$$

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&= \frac{e^4}{4s^2} \sum v_h(\mathbf{p_2})\bar{v}_a(\mathbf{p_2})\gamma^\mu_{ab} u_b(\mathbf{p_1})\bar{u}_g(\mathbf{p_1})\gamma^\nu_{gh} \\
&\quad u_f(\mathbf{p_3})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd} v_d(\mathbf{p_4})\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef} \\
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&\quad Tr(u(\mathbf{p_3})\bar{u}(\mathbf{p_3})\gamma_\mu v(\mathbf{p_4})\bar{v}(\mathbf{p_4})\gamma_\nu) \\
&= \frac{e^4}{4s^2} Tr(\not{\mathbf{p}}_2\gamma^\mu \; \not{\mathbf{p}}_1\gamma^\nu) Tr(\not{\mathbf{p}}_3\gamma_\mu \; \not{\mathbf{p}}_4\gamma_\nu)
\end{aligned}
$$

# Process calculation

## Formula

$$Tr(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} + g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta})$$

## Matrix Element

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{e^4}{4s^2}\sum[\bar{v}_a(\mathbf{p_2})\gamma^\mu_{ab}u_b(\mathbf{p_1})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd}v_d(\mathbf{p_4})] \\
&\quad [\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef}u_f(\mathbf{p_3})\bar{u}_g(\mathbf{p_1})\gamma^\nu_{gh}v_h(\mathbf{p_2})] \\
&= \frac{e^4}{4s^2}\sum v_h(\mathbf{p_2})\bar{v}_a(\mathbf{p_2})\gamma^\mu_{ab}u_b(\mathbf{p_1})\bar{u}_g(\mathbf{p_1})\gamma^\nu_{gh} \\
&\quad u_f(\mathbf{p_3})\bar{u}_c(\mathbf{p_3})\gamma_{\mu cd}v_d(\mathbf{p_4})\bar{v}_e(\mathbf{p_4})\gamma_{\nu ef} \\
&= \frac{e^4}{4s^2}\sum\nolimits_s Tr(v(\mathbf{p_2})\bar{v}(\mathbf{p_2})\gamma^\mu u(\mathbf{p_1})\bar{u}(\mathbf{p_1})\gamma^\nu) \\
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&\quad Tr(u(\mathbf{p_3})\bar{u}(\mathbf{p_3})\gamma_\mu v(\mathbf{p_4})\bar{v}(\mathbf{p_4})\gamma_\nu) \\
&= \frac{e^4}{4s^2}Tr(\not{\mathbf{p}}_2\gamma^\mu \not{\mathbf{p}}_1\gamma^\nu)Tr(\not{\mathbf{p}}_3\gamma_\mu \not{\mathbf{p}}_4\gamma_\nu) \\
&= \frac{8e^4}{s^2}[(\mathbf{p_1p_4})(\mathbf{p_2p_3}) + (\mathbf{p_1p_3})(\mathbf{p_2p_4})]
\end{aligned}$$

# Process calculation

## Formula

$$
\begin{aligned}
(\mathbf{p_1} - \mathbf{p_3})^2 &= \mathbf{p_1}^2 + \mathbf{p_3}^2 - 2\mathbf{p_1}\mathbf{p_3} \\
&= -2\mathbf{p_1}\mathbf{p_3} \\
&= -2\left(\frac{\sqrt{s}}{2}\frac{\sqrt{s}}{2} - \frac{\sqrt{s}}{2}\frac{\sqrt{s}}{2}\cos\theta\right) \\
(\mathbf{p_1} - \mathbf{p_4})^2 &= -2\frac{s}{4}(1+\cos\theta)
\end{aligned}
$$

## Differential Cross section

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= |\mathcal{M}|^2 \frac{1}{64\pi^2 s} \\
&= \frac{8e^4}{64\pi^2 s^3}[(\mathbf{p_1}\mathbf{p_4})(\mathbf{p_2}\mathbf{p_3}) + (\mathbf{p_1}\mathbf{p_3})(\mathbf{p_2}\mathbf{p_4})]
\end{aligned}
$$

# Process calculation

## Formula

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&= \frac{2\alpha^2}{s^3}[\tfrac{s}{4}(1+\cos\theta)\cdot\tfrac{s}{4}(1+\cos\theta) \\
&+ \tfrac{s}{4}(1-\cos\theta)\cdot\tfrac{s}{4}(1-\cos\theta)]
\end{aligned}
$$

# Process calculation

## Formula

$$
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&= \frac{2\alpha^2}{s^3}[\frac{s}{4}(1 + \cos\theta) \cdot \frac{s}{4}(1 + \cos\theta) \\
&+ \frac{s}{4}(1 - \cos\theta) \cdot \frac{s}{4}(1 - \cos\theta)] \\
&= \frac{2\alpha^2}{s^3}[\frac{s^2}{16}(1 + \cos\theta)^2 + \frac{s^2}{16}(1 - \cos\theta)^2]
\end{aligned}
$$

# Process calculation

## Formula

$$\begin{aligned}
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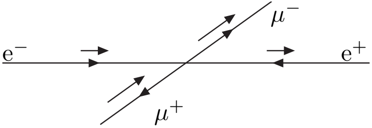
&= \frac{2\alpha^2}{s^3}[\frac{s}{4}(1+\cos\theta)\cdot\frac{s}{4}(1+\cos\theta) \\
&+ \frac{s}{4}(1-\cos\theta)\cdot\frac{s}{4}(1-\cos\theta)] \\
&= \frac{2\alpha^2}{s^3}[\frac{s^2}{16}(1+\cos\theta)^2+\frac{s^2}{16}(1-\cos\theta)^2] \\
&= \frac{\alpha^2}{4s}[1+\cos^2\theta]
\end{aligned}$$

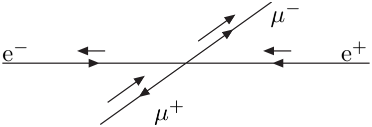
# Interpretation and experiments

$$\frac{d\sigma}{d\Omega} \sim (1-\cos\theta)^2 + (1+\cos\theta)^2$$

- Do the two terms have a particular meaning?
- Only the spin can lead to an angular distribution that is not flat
- Photon: Spin-1, mass zero $\rightarrow$ 2 dofs: $\pm 1$
- classical ED: 2 polarizations, no rest frame...

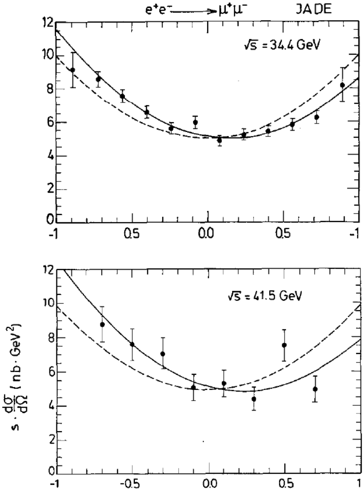


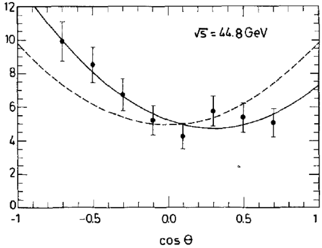

$$\begin{aligned} \theta(\mu^-, e^-) &= 0 \\ 1+\cos\theta &= 2 \quad \text{Probmax} \end{aligned}$$




$$\begin{aligned} \theta(\mu^-, e^-) &= 0 \\ 1-\cos\theta &= 0 \quad \text{Probmin} \end{aligned}$$

# Interpretation and experiments

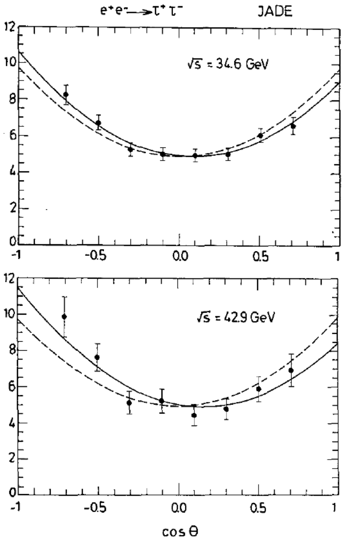


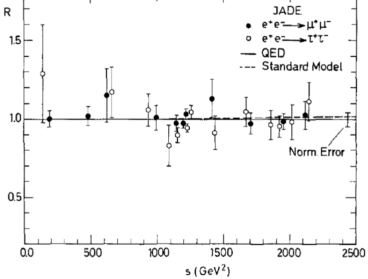




### $e^+e^- \to \mu^+\mu^-$

- JADE detector at PETRA
- $s \cdot \frac{d\sigma}{d\Omega}$ scale invariant
- low $s \to (1 + \cos^2\theta)$
- higher $s \to$ asymmetry not QED

# Interpretation and experiments




## $e^+e^- \to \tau^+\tau^-$

- Small mass dependence at high $\sqrt{s}$
- Lepton universality
- Agreement with QED

# The predictions

Bohr

$$\begin{aligned}
\vec{\mu} &= Current \cdot Surface \cdot \vec{n} \\
&= \frac{e}{t} \cdot \pi r^2 \cdot \vec{n} \\
&= \frac{e}{2\pi r/v} \cdot \pi r^2 \cdot \vec{n} \\
&= \frac{e}{2m}(mvr)\vec{n} \\
&= \frac{e}{2m}(\hbar\ell)\vec{n} \\
&= \mu_B \ell \vec{n} \\
\mu_B &= 5.8 \cdot 10^{-5} eV/T
\end{aligned}$$

Intrinsic magnetic moment:

$$\vec{\mu} = g \cdot \mu_B \cdot \vec{S}$$

**Definition**

$g$ is the gyromagnetic ratio, ratio of the magnetic dipole moment to the mechanical angular moment

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\end{aligned}
$$

Intrinsic magnetic moment:

$$\vec{\mu} = g \cdot \mu_B \cdot \vec{S}$$

Dirac

$$
\begin{aligned}
\vec{J} &= \vec{L} + \vec{S} \\
&= \vec{L} + \frac{1}{2}\vec{\sigma} \\
\vec{\mu} &= \frac{1}{2}\int \vec{x} \times \vec{j} \\
\vec{j} &= -e\bar{\psi}\vec{\gamma}\psi \\
\langle f|\vec{\mu}|f\rangle &\sim \frac{1}{2}\langle f|\vec{j}|f\rangle \\
&= \frac{-e}{2}\langle f|\bar{\psi}\vec{\gamma}\psi|f\rangle \\
&= \frac{-e}{2}\langle f|\vec{L} + \vec{\sigma}|f\rangle \\
&= \frac{-e}{2}\langle f|\vec{L} + g\vec{S}|f\rangle
\end{aligned}
$$

$$\vec{S} = 1/2\vec{\sigma}$$

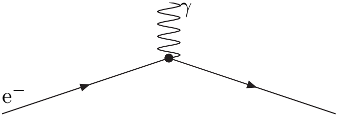
**Definition**

$g$ is the gyromagnetic ratio, ratio of the magnetic dipole moment to the mechanical angular moment

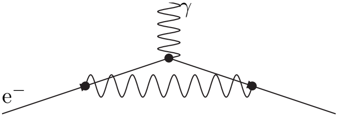
- The magnetic moment is anti-parallel with the Spin
- Dirac predicts $g = 2$!

# The predictions

## and QFT ?




Interaction with an external field: LO




Interaction with an external field: NLO

### Electromagnetic current

$$
\begin{aligned}
& -e\bar{u}\gamma^{\mu}u \\
= \quad & -\frac{e}{2m}\bar{u}[(p'+p)^{\mu} + i(p'-p)_{\nu}\frac{i}{2}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})]u \\
= \quad & -\frac{e}{2m}\bar{u}[(p'+p)^{\mu} + i(p'-p)_{\nu}\sigma^{\mu\nu}]u
\end{aligned}
$$

# The predictions

leads to:

$$\Delta\mu \sim \alpha/\pi \cdot \frac{e}{2m}$$
$$g = 2 + \alpha/\pi$$
$$a = \frac{g-2}{2} = \frac{1}{2}\frac{\alpha}{\pi} \sim 10^{-3}$$

| *Order* | *Diagrams* |
|---|---|
| 1 | 1 |
| 2 | 7 |
| 3 | 72 |
| 4 | 891 |
| 5 | 12672 |

**QED prediction $a_e$**

$$a_e = 1159652182.79 \cdot 10^{-12} \pm 7.79 \cdot 10^{-12}$$

8th order: Phys. Rev. Lett. 99, 110406 (2007)

# Measurements

## Electron Precession in B-field

$$
\begin{aligned}
mv_p^2/r &= ev_pB \\
mv_p/r &= eB \\
m\omega r/r &= eB \\
\omega_0 &= eB/m \\
m &\rightarrow m\gamma \\
\omega_C &= \omega_0/\gamma
\end{aligned}
$$

# Measurements

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## Spin Precession in B-field

Magnetic torque:
$\Delta E = g\mu_B B = \hbar\omega_L$
$\omega_L = g(eB)/(2m) = \frac{1}{2}g\omega_0$
Relativistic corrections (Thomas):
$\omega_P = \omega_L - \omega_T = \frac{g}{2}\omega_0 - \frac{\gamma-1}{\gamma}\omega_0$

# Measurements

## Electron Precession in B-field

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mv_p^2/r &= ev_pB \\
mv_p/r &= eB \\
m\omega r/r &= eB \\
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$$

## Spin Precession in B-field

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## Phase difference

$\Delta\omega = \omega_L - \omega_0 = a_e\omega_0$
Relativistic: $\Delta\omega = \omega_P - \omega_C = a_e\omega_0$

# Measurements

## Electron Precession in B-field

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m &\rightarrow m\gamma \\
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\end{aligned}
$$

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Relativistic corrections (Thomas):
$\omega_P = \omega_L - \omega_T = \frac{g}{2}\omega_0 - \frac{\gamma-1}{\gamma}\omega_0$

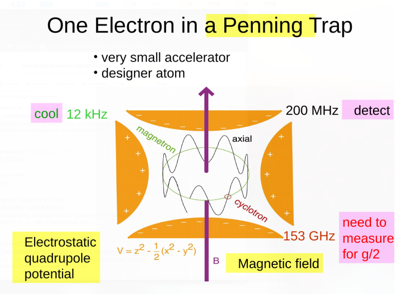
## Phase difference

$\Delta\omega = \omega_L - \omega_0 = a_e\omega_0$
Relativistic: $\Delta\omega = \omega_P - \omega_C = a_e\omega_0$

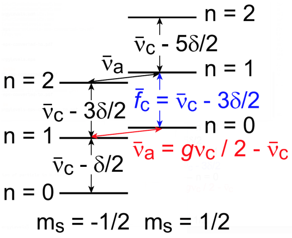
## $a_e$

$a_e = 0$ : Spin in phase with electron rotation
$a_e \neq 0$ : Spin precession not in phase with precession of particle in B-field

# Measurements




- Penning trap electrons (small scale experiment)
- $\delta/\nu_C$: relativistic shift
- f Cyclotron : 149 GHz
- f Anomaly : 173 MHZ

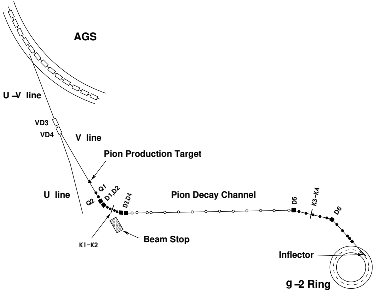



**$a_e$**

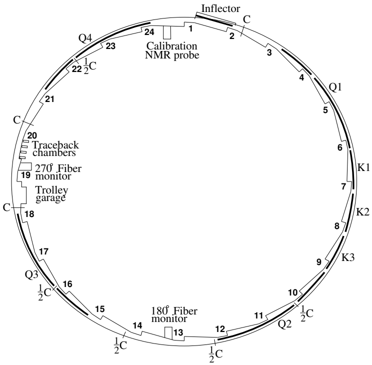
$$a_e = 115965218073(28) \cdot 10^{-14}$$
$$\alpha^{-1} = 137.035999084(51)$$

- test QED to $10^{-13}$
- determine $\alpha$ to 0.37ppb ($\approx 10^{-9}$)
- natural scale: $m_e \approx 0.5 MeV$

# Measurements

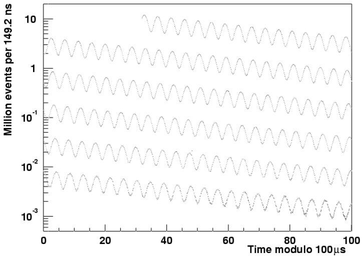



- muon liftime penning trap not feasible
- 24GeV protons to produce pions which decay to muons
- muons decay to electrons

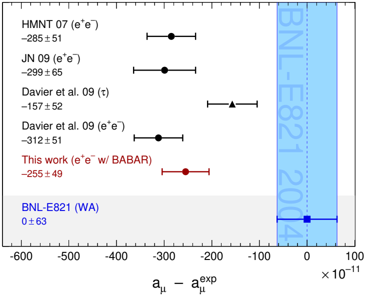



- calorimeters detect the electrons
- excellent knowledge of B-field necessary

# Measurements




- electron counting rate varies as function of the precession of the spin
- natural scale of experiment $m_\mu \approx 0.105 GeV$




- Hadronic contribution (non QED) important (695)
- Prediction is mixture of calculation and measurement
- Supersymmetry?