Phase space - Dalitz diagram

We consider the transition $i \to f$ where the state f contains n particules of momenta $p_1, p_2, \dots p_n$. The "phase space for n particles" is defined by

$$\begin{aligned} R_n(\vec{P}, E) = \int (2\pi)\delta(\sum_{j=1}^{n} E_j - E) \cdot (2\pi)^3\delta^3(\sum_{j=1}^{n} \vec{P}_j - \vec{P}) \\ \times \prod_{j=1}^{n}[(2\pi)\delta(P_j^2 - m_j^2)] \cdot \prod_{j=1}^{n} \frac{d^4P_j}{(2\pi)^4} \end{aligned} \tag{1}$$

The $i \to f$ transition probability can be expressed by the square of the matrix elements of the transition between the two states, integrated over the allowed kinematic domain.

$$\begin{aligned} \text{Prob}(i \to f) = \int |\langle f|T|i\rangle|^2 \cdot (2\pi)\delta(\sum_{j=1}^{n} E_j - E) \cdot (2\pi)^3\delta^3(\sum_{j=1}^{n} \vec{P}_j - \vec{P}) \\ \times \prod_{j=1}^{n}[(2\pi)\delta(P_j^2 - m_j^2)] \cdot \prod_{j=1}^{n} \frac{d^4P_j}{(2\pi)^4} \end{aligned} \tag{2}$$

If the matrix elements do not depend on the P_j terms, Prob $(i \to f) \sim R_n$. Some transitions are disfavoured with respect to others, not because of the dynamics but because of the space space allowed for the transition. Moreover, the phase space constrains the kinematic of the particles in the final state. For example, in the case of the decay of a particle 0 in three particles 1,2,3 for which the matrix elements are independent from P_1, P_2, P_3, the m_{12} invariant mass distribution is expressed by

$$\frac{dN}{dm_{12}} \sim \frac{dR_3}{dm_{12}}. \tag{3}$$

dR_3/dm_{12} must be evaluated to determine any possible variation of $|\langle f|T|i\rangle|$ with m_{12} (in the case of a resonance, for example).

1) Show that

$$R_n(P, E) = \int (2\pi)\delta(\sum_{j=1}^{n} E_j - E) \cdot (2\pi)^3\delta^3(\sum_{j=1}^{n} \vec{P}_j - \vec{P}) \prod_{j=1}^{n}[\frac{d^3P_j}{(2\pi)^3 2E_j}] \tag{4}$$

where $E_j = \sqrt{\vec{P}_j^2 + m_j^2}$

2) Evaluate the phase space for two particles in the centre of mass frame, $R_2(0, E^*)$.

3) Compare the phase space for the transitions $K^- \to \mu^-\nu_\mu$ and $K^- \to e^-\nu_e$. The corresponding branching ratios are $(63.51 \pm 0.16)\%$ and $(1.54 \pm 0.07) \cdot 10^{-5}$ respectively. What do you conclude ?

4) Using the recurrence relation

$$R_n(0,E) = \int \frac{d^3P_n}{(2\pi)^3 2E_n} \cdot R_{n-1}(0, m_{n-1}), \tag{5}$$

where m_{n-1} is the invariant mass of the set of particles 1,2,3,. . . ,n-1, show that

$$\left| \frac{dR_3}{dm_{12}^2} \right| = \frac{|\vec{P}_3|}{8\pi^2 E^*} \cdot R_2(0, m_{12}). \tag{6}$$

6) Evaluate $dR_3/d(m_{12}^2)$ for the ω decay in three photons.

7) For a system of three particles in the centre of mass frame,

$$dR_3(0, E^*) = \frac{1}{32\pi^3} dE_1 \cdot dE_2. \tag{7}$$

If the matrix elements of a transition to a 3 body final state is independent from the momenta of those particles, how are distributed the events in the (m_{12}^2, m_{23}^2) plane ?