

3. Isolated inter.

$$H_{ext} = H_{c1}$$

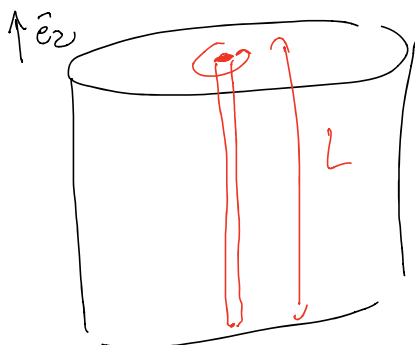
$$G_S \Big|_{no \text{ flux}} = G_S \Big|_{first \text{ inter.}}$$

$$1. \quad G_S \Big|_{no \text{ flux}} = F_S(T, B=0) \quad (\text{in absence of flux, } B=0)$$

$$G_S \Big|_{inter} = F_S(T, B=0) - H \int d\vec{r} \cdot \vec{B}(\vec{r}) + \varepsilon_1 \cdot L$$

$$\Rightarrow \quad \cancel{F_S(T, B=0)} - H_{c1} \underbrace{\int d\vec{r} \cdot \vec{B}(\vec{r})}_{\Phi_0 \cdot L} + \varepsilon_1 \cdot L = \cancel{F_S(T, B=0)}$$

$$\Phi = \iint d\vec{S} \cdot \vec{B} \quad \Rightarrow \quad \boxed{H_{c1} = \frac{\varepsilon_1}{\Phi_0}}$$



$$\vec{B} = B(r) \vec{e}_z$$

$$\psi(\vec{r}) = \psi(r, \phi) = |\psi_0| f(r) e^{i\phi}$$

the angle ϕ

Before:

$$\psi(\vec{r}) = |\psi(\vec{r})| \cdot e^{i\theta(\vec{r})}$$

$$\vec{A} = A(r) \cdot \vec{e}_\phi$$

2. $A(r) \approx \frac{B(0) \cdot r}{2}$ near the center.

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r} \frac{d(r \cdot A(r))}{dr} \cdot \vec{e}_z$$

$$r B(r) = \frac{d(r \cdot A(r))}{dr}$$

$$A(r) = \frac{1}{r} \cdot \int_0^r dr' r' B(r')$$

near the center of the wire, where the magnetic field is roughly constant $B(0)$

$$A(r) \approx \frac{1}{r} \cdot B(0) \cdot \int_0^r dr' r'$$

$$= \frac{B(0) r}{2}$$

$$\Rightarrow \boxed{A(r) \approx \frac{B(0) r}{2}}$$

far away from the wire: $B(r) = 0$

$$A(r) \propto \frac{1}{r}$$

total flux: $\iint \vec{B} \cdot d\vec{S} = \Phi$

$$\oint \vec{A}(r) \cdot d\vec{\ell} = \Phi$$

$$A(r) \cdot 2\pi r = \Phi$$

$$\boxed{A(r) = \frac{\Phi}{2\pi r}}$$

$$4. \quad r \rightarrow 0$$

$$f = c \cdot r^n \quad n \geq 1.$$

$$f - f^3 - \xi^2 \left(\left(\frac{1}{2} - \frac{2\pi A}{\xi} \right)^2 f - \frac{1}{2} \frac{d}{dr} \left(r \frac{df}{dr} \right) \right) = 0$$

$$c r^n - c^3 r^{3n} - \xi^2 \left\{ \left(\frac{1}{2} - \frac{\pi B(0) \cdot r}{\xi} \right)^2 c r^n - \underline{c n^2 r^{n-2}} \right\} = 0$$

$$\text{leading terms: } c r^{n-2} - c n^2 r^{n-2} = 0$$

$$\Rightarrow (1 - n^2) = 0$$

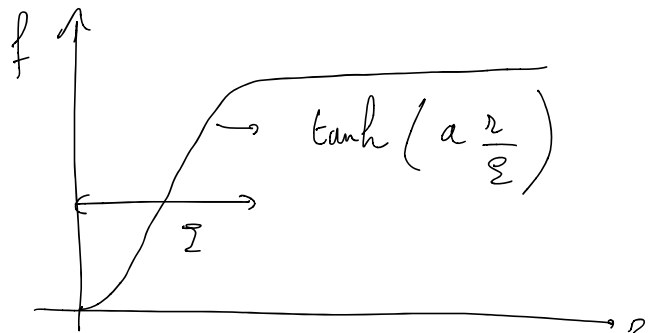
$$\Rightarrow \boxed{n = 1}$$

only odd terms
will remain in the
expansion:

$$f = c r + d r^3$$

$$c \sim \frac{1}{2\xi}$$

f will start to saturate when $r \gtrsim \xi$



5. $\vec{A} \Rightarrow \vec{\Sigma}$

$$\vec{J} = \frac{e\hbar}{m^*} |\psi_\infty|^2 \oint \left(\frac{1}{2} - \frac{2\pi A(r)}{\Phi_0} \right) \vec{e}_\phi$$

$f \rightarrow 0$ at the core

But $\vec{A} \Rightarrow \vec{\Sigma}$, and we ignore the core here by taking $f=1$ everywhere.

$$\begin{aligned} \vec{\nabla}_\times \vec{J} &= \frac{e\hbar}{m^*} |\psi_\infty|^2 \vec{\nabla}_\times \left[\frac{1}{2} - \frac{2\pi A(r)}{\Phi_0} \right] \vec{e}_\phi \\ &= \frac{e\hbar}{m^*} |\psi_\infty|^2 \left\{ \underbrace{\vec{\nabla}_\times \left[\frac{1}{2} \cdot \vec{e}_\phi \right]}_{?} - \frac{e\hbar}{\hbar} \vec{B}(\vec{r}) \right\} \end{aligned}$$

$$\iint d\vec{S} \left\{ \vec{\nabla}_\times \left(\overset{\text{arbitrary function}}{g(\vec{r})} \frac{\vec{e}_\phi}{2} \right) \right\} = \oint_{\Gamma} \left(g(\vec{r}) \cdot \frac{\vec{e}_\phi}{2} \right) d\vec{e} = 0$$

$$\iint d\vec{S} \left\{ (\vec{\nabla} g) \times \frac{\vec{e}_\phi}{2} + g(\vec{r}) \cdot \vec{\nabla}_\times \left(\frac{\vec{e}_\phi}{2} \right) \right\} = 0$$

$$= \int d\phi \int dr \cdot \cancel{\vec{\nabla} g \times \frac{\vec{e}_\phi}{2}} + \underbrace{\iint d\vec{S} g(\vec{r}) \cdot \vec{\nabla}_\times \left(\frac{\vec{e}_\phi}{2} \right)}_{\text{I}} = 0$$

$$\Rightarrow \text{I} = \int d\phi \int dr \left(\vec{e}_\phi \times \vec{\nabla} g \right) \cdot \underline{\vec{e}_z}$$

$$= \int d\phi \int dr \left(\vec{e}_\phi \times \left[\frac{\partial g}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial g}{\partial \phi} \vec{e}_\phi + \frac{\partial g}{\partial z} \vec{e}_z \right] \cdot \vec{e}_z \right)$$

$$\begin{aligned}
&= \int d\phi \int dr \left(-\bar{e}_z \frac{\partial g}{\partial r} + \frac{\partial g}{\partial r} \bar{e}_z \right) \cdot \bar{e}_z \\
&= - \int d\phi \, dr \, \frac{\partial g}{\partial r} = \int d\phi \, g(r=0) \\
&= 2\pi \, g(r)
\end{aligned}$$

For any g :

$$\begin{aligned}
\iint d\vec{S} \left\{ g(\vec{r}) \cdot \vec{\nabla} \times \left(\frac{\vec{e}_\phi}{r} \right) \right\} &= 2\pi \, g(r=0) \\
\Rightarrow \vec{\nabla} \times \left(\frac{\vec{e}_\phi}{r} \right) &= \bar{e}_z \cdot 2\pi \delta^{(2)}(\vec{r})
\end{aligned}$$

$$\vec{\nabla} \times \vec{J} = \frac{e_* \hbar}{m_*} |\psi_\infty|^2 \left[2\pi \delta^{(2)}(\vec{r}) \cdot \bar{e}_z - \frac{e_*}{\hbar} \vec{B}(\vec{r}) \right]$$

$$\vec{\nabla} \times \vec{J} + \frac{e_*^2 n_s}{m_*} \vec{B} = \frac{e_* \hbar}{m_*} n_s \delta^{(2)}(\vec{r}) \cdot \bar{e}_z$$

$$\int_0^L dL \, \vec{\nabla} \times \vec{J} + \vec{B} = \underbrace{\vec{E}_0 \cdot \delta^{(2)}(\vec{r}) \cdot \bar{e}_z}_{\text{...}}$$

$$\int_0^L dL = \frac{m_*}{e_*^2 n_s}$$

$$6. \quad \vec{\nabla} \times \vec{B} = \int_0 \vec{J}$$

$$\Rightarrow \underbrace{L^2 \vec{\nabla} \times (\vec{\nabla} \times \vec{B})}_{-L^2 \nabla^2 \vec{B}} + \vec{B} = \bar{e}_z \vec{E}_0 \delta^{(2)}(\vec{r})$$

$$\Rightarrow \vec{\nabla}^2 \vec{B} - \frac{\vec{B}}{L^2} = -\frac{\vec{E}_0}{L^2} \bar{e}_z \cdot \delta^{(2)}(\vec{r})$$

$$\alpha = \frac{1}{\lambda} \quad \left\{ \begin{array}{l} \Rightarrow \text{exact solution} \quad B(r) = \frac{1}{2\pi} \frac{\Phi_0}{\lambda^2} K_0\left(\frac{r}{\lambda}\right) \\ \beta = \frac{\Phi_0}{\lambda^2} \end{array} \right.$$

$$K_0\left(\frac{r}{\lambda}\right) \approx \sqrt{\frac{\pi\lambda}{2r}} e^{-\frac{r}{\lambda}} \quad r \gg \lambda$$

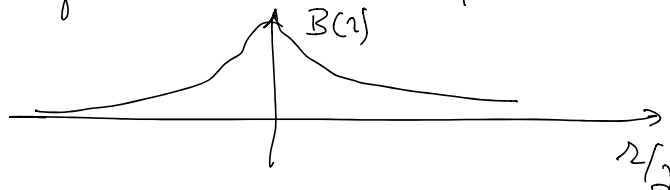
$$K_0\left(\frac{r}{\lambda}\right) \approx \ln\left(\frac{\lambda}{r}\right) + 0.18 \quad r \ll \lambda$$

$$B(r) \approx \frac{\Phi_0}{2\pi\lambda^2} \cdot \sqrt{\frac{\pi\lambda}{2r}} e^{-r/\lambda} \quad r \gg \lambda$$

$$B(r) \approx \frac{\Phi_0}{2\pi\lambda^2} \left[\ln\left(\frac{\lambda}{r}\right) + 0.12 \right] \quad \xi \leq r \ll \lambda$$

for small r , the divergence is cut off at $r \sim \xi$
 where $|\psi|^2$ starts dropping to zero.

$\Rightarrow B(r)$ is regular at the center of the vortex.



7. $H_{c1} = \frac{\mathcal{E}_1}{\Phi_0}$ free energy per unit length of the vortex line.

$$\mathcal{E}_1 = \frac{1}{2\lambda} \iint_{r > \xi} dS \left(B^2 + \lambda_L^2 (\nabla \times \vec{B})^2 \right)$$

$$\vec{C} \rightarrow \vec{B}$$

$$\vec{D} \rightarrow \vec{\nabla} \times \vec{B}$$

$$\vec{\nabla} \cdot (\vec{B} \times (\vec{\nabla} \times \vec{B})) = (\vec{\nabla} \times \vec{B})^2 - \vec{B} \cdot (\vec{\nabla} \times (\vec{\nabla} \times \vec{B}))$$

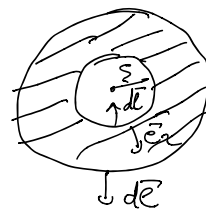
$$\varepsilon_1 = \frac{1}{2\mu_0} \iint_{\Sigma \cup \bar{\Sigma}} dS \left\{ B^2 + \lambda^2 \vec{\nabla} \cdot (\vec{B} \times (\vec{\nabla} \times \vec{B})) + \lambda^2 \vec{B} \cdot (\vec{\nabla} \times (\vec{\nabla} \times \vec{B})) \right\}$$

$$= \frac{1}{2\mu_0} \iint dS \left(\vec{B} + \lambda^2 \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) \right) \cdot \vec{B}$$

$$+ \frac{\lambda^2}{2\mu_0} \oint d\vec{\ell} \cdot (\vec{B} \times (\vec{\nabla} \times \vec{B}))$$

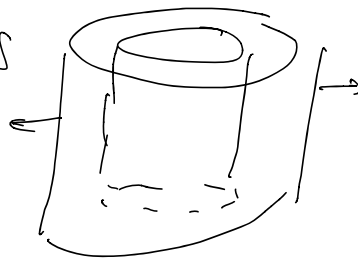
around the inner and outer perimeter of the integration area

$$= \frac{1}{2\mu_0} \iint_{\Sigma \cup \bar{\Sigma}} dS \cdot \vec{B}_0 \cdot g^{(1)}(\vec{r}) \cdot \vec{e}_z + \text{second term}$$



$$\iiint \vec{\nabla} \cdot \vec{F} dV$$

$$= \iint \vec{F} \cdot \vec{n} \cdot dS$$



$$\begin{cases} \vec{\nabla} \times \vec{B} = -\frac{dB(r)}{dr} \vec{e}_\phi \\ \vec{B} \times (\vec{\nabla} \times \vec{B}) = B \cdot \frac{dB}{dr} \cdot \vec{e}_r \end{cases}$$

$$= -\frac{\lambda^2}{2\mu_0} \cdot B \cdot \frac{dB}{dr} \cdot 2\pi r \bigg|_{r=\Sigma} + \dots$$

~~$r = \infty$~~

$$= -\frac{A^2}{2\mu_0} 2\pi \Sigma B(\Sigma) B'(\Sigma)$$

$$= -\frac{A^2}{2\mu_0} 2\pi \Sigma B(\Sigma) \cdot \frac{\Phi_0}{2\pi A^2} \cdot \frac{\Sigma}{A} \left(-\frac{A}{\Sigma}\right)$$

$$= \left[\frac{B(\Sigma) \Phi_0}{2\mu_0} = \varepsilon_1 \right]$$

$$\varepsilon_1 \approx \frac{\Phi_0}{2\mu_0} \cdot \frac{\Phi_0}{2\pi A^2} \left(\ln \left(\frac{A}{\Sigma} \right) + 0.12 \right)$$

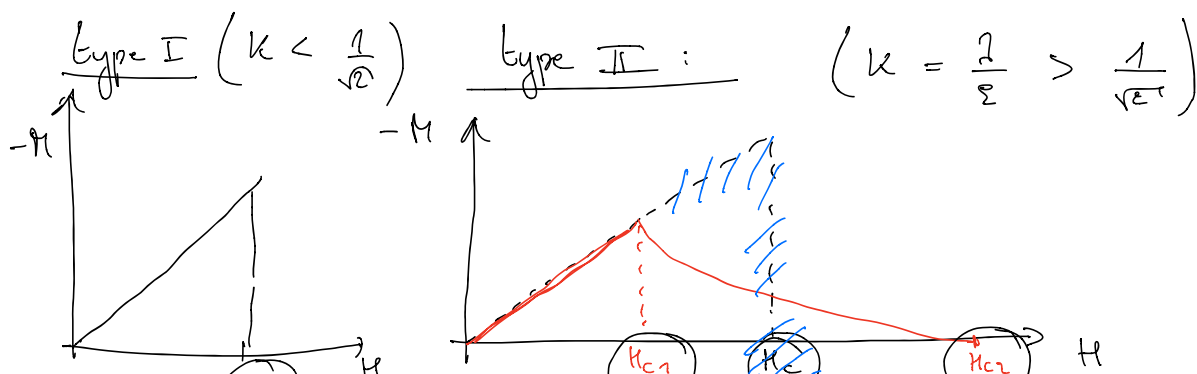
$$\boxed{\varepsilon_1 = \frac{\Phi_0^2}{2\mu_0 2\pi A^2} \ln \kappa}$$

$A \gg \Sigma$ (London)

$$\kappa = \frac{A}{\Sigma}$$

$$8. H_C = \frac{1}{\sqrt{2}} \int_0^{\frac{h}{2\mu_0 c \kappa}} \Sigma \lambda \cdot c \kappa \quad (\text{see calculation of } H_{C2})$$

$$\frac{H_{C1}}{H_C} = \frac{\frac{\Phi_0}{2\mu_0 2\pi A^2} \ln \kappa \cdot \frac{\sqrt{2}}{2} \Sigma \lambda \cdot c \kappa}{\frac{h}{2\mu_0 c \kappa}} = \left[\frac{1}{\sqrt{2}} \frac{\ln \kappa}{\kappa} = \frac{H_{C1}}{H_C} \right] (A \gg \Sigma)$$



(H_c) "

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$$\frac{H_{c_2}}{H_c} = \sqrt{2} \cdot \kappa$$

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