

TD 2 – Ginzburg–Landau Theory : correction

1. Gauge invariance

Maxwell equations and electromagnetic potentials

We start with Maxwell's equations in SI units

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}} \quad (1)$$

$$\boxed{\nabla \cdot \mathbf{B} = 0} \quad (2)$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \quad (3)$$

and

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}. \quad (4)$$

The second equation $\nabla \cdot \mathbf{B} = 0$ implies that the magnetic field can be written in terms of a vector potential $\mathbf{A}$,

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (5)$$

Since $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, this representation automatically satisfies $\nabla \cdot \mathbf{B} = 0$.

Using Faraday's law (Eq.(3)) and substituting $\mathbf{B} = \nabla \times \mathbf{A}$, we obtain

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{A}) \quad (6)$$

$$= -\nabla \times \frac{\partial \mathbf{A}}{\partial t} \quad (7)$$

Hence

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \quad (8)$$

A vector field with zero curl can be written as the gradient of a scalar. We therefore introduce the scalar potential V such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \quad (9)$$

Thus

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad (10)$$

The electromagnetic fields are therefore expressed in terms of the potentials as

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}.} \quad (11)$$

The physical electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are uniquely measurable, whereas the electromagnetic potentials V and $\mathbf{A}$ are not uniquely defined. This non-uniqueness is the origin of gauge invariance.

Let's introduce an arbitrary scalar function $\chi(\mathbf{r}, t)$ and consider the gauge transformation

$$\mathbf{A}' = \mathbf{A} + \nabla \chi \quad (12)$$

and

$$V' = V - \frac{\partial \chi}{\partial t}. \quad (13)$$

We can easily verify that the physical electromagnetic fields are unchanged. For the magnetic field,

$$\mathbf{B}' = \nabla \times \mathbf{A}' \quad (14)$$

$$= \nabla \times (\mathbf{A} + \nabla \chi) \quad (15)$$

$$= \nabla \times \mathbf{A} + \nabla \times \nabla \chi. \quad (16)$$

Since $\nabla \times \nabla \chi = 0$, we obtain $\mathbf{B}' = \mathbf{B}$.

For the electric field,

$$\mathbf{E}' = -\nabla V' - \frac{\partial \mathbf{A}'}{\partial t} \quad (17)$$

$$= -\nabla \left(V - \frac{\partial \chi}{\partial t} \right) - \frac{\partial}{\partial t} (\mathbf{A} + \nabla \chi) \quad (18)$$

$$= -\nabla V + \nabla \frac{\partial \chi}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial}{\partial t} \nabla \chi. \quad (19)$$

Since we can exchange the position of the time derivative and the nabla, $\nabla \frac{\partial \chi}{\partial t} = \frac{\partial}{\partial t} \nabla \chi$, the two additional terms cancel, and therefore $\mathbf{E}' = \mathbf{E}$.

Thus the electromagnetic fields are invariant under the gauge transformation.

Transformation of the GL order parameter

The first GL equation is

$$\alpha \psi + \beta |\psi|^2 \psi + \frac{1}{2m^*} \left(\frac{\hbar}{i} \nabla - e^* \mathbf{A} \right)^2 \psi = 0. \quad (20)$$

The only potentially problematic term under a gauge transformation is the kinetic term, $\left(\frac{\hbar}{i} \nabla - e^* \mathbf{A} \right) \psi$ since if one changes $\mathbf{A}$ without changing ψ , the GL equation is not invariant. Therefore the order parameter must be transformed accordingly

We look for a transformation of the form

$$\boxed{\psi' = e^{ie^* \chi / \hbar} \psi.} \quad (21)$$

We can easily check that this is the correct transformation.

Using $\mathbf{A}' = \mathbf{A} + \nabla \chi$, we can write

$$\begin{aligned}
& (-i\hbar\nabla - e^* \mathbf{A}') \psi' \\
& = [-i\hbar\nabla - e^*(\mathbf{A} + \nabla\chi)] (e^{ie^*\chi/\hbar}\psi).
\end{aligned} \tag{22}$$

With

$$\nabla (e^{ie^*\chi/\hbar}\psi) = e^{ie^*\chi/\hbar}\nabla\psi + \psi\nabla (e^{ie^*\chi/\hbar}) \tag{23}$$

$$= e^{ie^*\chi/\hbar} \left[\nabla\psi + i\frac{e^*}{\hbar}(\nabla\chi)\psi \right]. \tag{24}$$

The first term gives

$$-i\hbar\nabla\psi' = e^{ie^*\chi/\hbar} [-i\hbar\nabla\psi + e^*(\nabla\chi)\psi]. \tag{25}$$

The vector-potential term gives

$$-e^* \mathbf{A}' \psi' = -e^{ie^*\chi/\hbar} e^* (\mathbf{A} + \nabla\chi) \psi. \tag{26}$$

Combining both contributions,

$$\begin{aligned}
& (-i\hbar\nabla - e^* \mathbf{A}') \psi' \\
& = e^{ie^*\chi/\hbar} [-i\hbar\nabla\psi + e^*(\nabla\chi)\psi - e^* \mathbf{A}\psi - e^*(\nabla\chi)\psi].
\end{aligned} \tag{27}$$

The terms proportional to $\nabla\chi$ cancel, giving

$$(-i\hbar\nabla - e^* \mathbf{A}') \psi' = e^{ie^*\chi/\hbar} (-i\hbar\nabla - e^* \mathbf{A}) \psi. \tag{28}$$

Taking the modulus squared, we obtain the kinetic term

$$|(-i\hbar\nabla - e^* \mathbf{A}') \psi'|^2 = |(-i\hbar\nabla - e^* \mathbf{A}) \psi|^2, \tag{29}$$

Furthermore,

$$|\psi'|^2 = |\psi|^2. \tag{30}$$

Therefore all the terms in the GL free energy are gauge invariant.

Transformation of the phase and superfluid velocity

Writing

$$\psi = |\psi|e^{i\theta}, \tag{31}$$

the transformation $\psi' = e^{ie^*\chi/\hbar}\psi$ implies

$$\boxed{\theta' = \theta + \frac{e^*}{\hbar}\chi.} \tag{32}$$

The superfluid velocity is given by

$$m^* \mathbf{v}_s = \hbar\nabla\theta - e^* \mathbf{A}. \tag{33}$$

Under the gauge transformation,

$$\hbar\nabla\theta' - e^*\mathbf{A}' = \hbar\nabla\left(\theta + \frac{e^*}{\hbar}\chi\right) - e^*(\mathbf{A} + \nabla\chi) \quad (34)$$

$$= \hbar\nabla\theta + e^*\nabla\chi - e^*\mathbf{A} - e^*\nabla\chi \quad (35)$$

$$= \hbar\nabla\theta - e^*\mathbf{A} \quad (36)$$

Therefore $\mathbf{v}'_s = \mathbf{v}_s$, which shows that the superfluid velocity is gauge invariant.

In fact, the combination $\hbar\nabla\theta - e^*\mathbf{A}$ is gauge invariant, even though θ and $\mathbf{A}$ separately are gauge dependent.

Final result

The local gauge transformation compatible with the GL equations is therefore

$$\boxed{\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla\chi, \\ V' &= V - \frac{\partial\chi}{\partial t}, \\ \psi' &= e^{ie^*\chi/\hbar}\psi, \\ \theta' &= \theta + \frac{e^*}{\hbar}\chi \end{aligned}} \quad (37)$$

With these transformations, the electromagnetic fields, the GL free energy, the GL equations, and the superfluid velocity remain invariant.

2. The London limit

In the London limit, the amplitude of the superconducting order parameter is assumed to be constant $|\psi(\mathbf{r})| = |\psi_\infty|$ while its phase $\theta(\mathbf{r})$ may vary in space.

We write

$$\psi(\mathbf{r}) = |\psi_\infty|e^{i\theta(\mathbf{r})} \quad (38)$$

The Ginzburg–Landau free-energy density is

$$f_s = f_{n0} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*} \left| \left(\frac{\hbar}{i}\nabla - e^*\mathbf{A} \right) \psi \right|^2 + \frac{B^2}{2\mu_0} \quad (39)$$

2.1 Bulk value of the order parameter

For a bulk superconductor, deep inside the sample and in the absence of fields and gradients,

$$\nabla\psi = 0, \quad \mathbf{A} = 0, \quad B = 0. \quad (40)$$

The free-energy density reduces to

$$f_s = f_{n0} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4. \quad (41)$$

Let $x = |\psi|^2$, then $f_s - f_{n0} = \alpha x + \frac{\beta}{2}x^2$.

At equilibrium,

$$\frac{df_s}{dx} = 0, \quad (42)$$

which gives $\alpha + \beta x = 0$ and therefore $|\psi_\infty|^2 = -\frac{\alpha}{\beta}$, and $|\psi_\infty| = \sqrt{-\frac{\alpha}{\beta}}$ (with $\alpha < 0$ and $\beta > 0$).

The corresponding condensation free-energy density is $f_s - f_{n0} = \frac{\alpha^2}{\beta^2}$.
Hence

$$\boxed{f_s - f_{n0} = -\frac{\alpha^2}{2\beta}}. \quad (43)$$

2.2 Phase stiffness and total free energy

We now assume that $\psi(\mathbf{r}) = |\psi_\infty|e^{i\theta(\mathbf{r})}$ with constant amplitude but slowly varying phase.

Since $\nabla\psi = i\psi\nabla\theta$ we obtain $\left(\frac{\hbar}{i}\nabla - e^*\mathbf{A}\right)\psi = (\hbar\nabla\theta - e^*\mathbf{A})\psi$.

Therefore the kinetic contribution becomes

$$\frac{1}{2m^*} \left| \left(\frac{\hbar}{i}\nabla - e^*\mathbf{A} \right) \psi \right|^2 = \frac{\hbar^2 |\psi_\infty|^2}{2m^*} \left(\nabla\theta - \frac{e^*}{\hbar}\mathbf{A} \right)^2. \quad (44)$$

Using the definition given for the superfluid stiffness $\rho_s = \frac{\hbar^2}{2m^*} |\psi_\infty|^2$, the kinetic term becomes

$$f_{\text{kin}} = \rho_s \left(\nabla\theta - \frac{e^*}{\hbar}\mathbf{A} \right)^2 \quad (45)$$

Since the amplitude is fixed at its equilibrium value,

$$\alpha |\psi_\infty|^2 + \frac{\beta}{2} |\psi_\infty|^4 = -\frac{\alpha^2}{2\beta}. \quad (46)$$

Thus

$$\boxed{F_s = \int d^3r \left[f_{n0} - \frac{\alpha^2}{2\beta} + \rho_s \left(\nabla\theta - \frac{e^*}{\hbar}\mathbf{A} \right)^2 + \frac{B^2}{2\mu_0} \right]}. \quad (47)$$

The coefficient ρ_s measures the energetic cost of spatial variations of the phase and is therefore called the superfluid stiffness.

2.3 Superconducting current density

We define the superconducting current through the dependence of the condensate free energy on the vector potential,

$$\mathbf{J}_s = -\frac{\delta F_s[\mathbf{A}]}{\delta \mathbf{A}} \quad (48)$$

The relevant $\mathbf{A}$ -dependent condensate term is

$$\rho_s \left(\nabla\theta - \frac{e^*}{\hbar}\mathbf{A} \right)^2 \quad (49)$$

Taking the functional derivative gives

$$\mathbf{J}_s = 2\rho_s \frac{e^*}{\hbar} \left(\nabla\theta - \frac{e^*}{\hbar}\mathbf{A} \right) \quad (50)$$

$$= \frac{e^* |\psi_\infty|^2}{m^*} (\hbar\nabla\theta - e^*\mathbf{A}). \quad (51)$$

Hence

$$\mathbf{J}_s = \frac{e^*|\psi_\infty|^2}{m^*} (\hbar\nabla\theta - e^*\mathbf{A}) \quad (52)$$

Using the definition of the superfluid stiffness $m^*\mathbf{v}_s = \hbar\nabla\theta - e^*\mathbf{A}$, we recover

$$\boxed{\mathbf{J}_s = e^*|\psi_\infty|^2\mathbf{v}_s} \quad (53)$$

2.4 Second London equation and penetration depth

Taking the curl of the current density,

$$\nabla \times \mathbf{J}_s = \frac{e^*|\psi_\infty|^2}{m^*} \nabla \times (\hbar\nabla\theta - e^*\mathbf{A}). \quad (54)$$

As $\nabla \times \nabla\theta = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$, we obtain

$$\nabla \times \mathbf{J}_s = -\frac{e^{*2}|\psi_\infty|^2}{m^*} \mathbf{B} \quad (55)$$

We define the London penetration depth by

$$\lambda_L^2 = \frac{m^*}{\mu_0 e^{*2} |\psi_\infty|^2} \quad (56)$$

Then the second London equation becomes

$$\boxed{\nabla \times \mathbf{J}_s = -\frac{1}{\mu_0 \lambda_L^2} \mathbf{B}.} \quad (57)$$

We can also write $\lambda_L^2 = -\frac{m^*\beta}{\mu_0 e^{*2}\alpha}$.

To obtain the field penetration law, we use the Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s. \quad (58)$$

Taking the curl,

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \nabla \times \mathbf{J}_s \quad (59)$$

Using

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \quad (60)$$

and $\nabla \cdot \mathbf{B} = 0$, we get

$$-\nabla^2 \mathbf{B} = -\frac{1}{\lambda_L^2} \mathbf{B}. \quad (61)$$

Hence

$$\boxed{\nabla^2 \mathbf{B} = \frac{\mathbf{B}}{\lambda_L^2}.} \quad (62)$$

For a planar surface with the superconductor occupying $x > 0$,

$$B(x) = B(0)e^{-x/\lambda_L}. \quad (63)$$

Thus λ_L is the characteristic penetration depth of the magnetic induction into the superconductor.

2.5 Critical current density in the London approximation

The condensation free-energy density is

The thermodynamic critical field is defined by equating this condensation energy to the magnetic energy density corresponding to the critical field,

$$\boxed{\frac{\mu_0 H_c^2}{2} = \frac{\alpha^2}{2\beta}} \quad (64)$$

A supercurrent has kinetic-energy density

$$f_{\text{kin}} = \frac{1}{2} m^* |\psi_\infty|^2 v_s^2. \quad (65)$$

Since

$$J_s = e^* |\psi_\infty|^2 v_s, \quad (66)$$

we obtain

$$f_{\text{kin}} = \frac{m^*}{2e^{*2} |\psi_\infty|^2} J_s^2 = \frac{\mu_0 \lambda_L^2}{2} J_s^2 \quad (67)$$

Within the London approximation, the critical current density is obtained when the kinetic-energy cost equals the condensation-energy gain,

$$\frac{\mu_0 \lambda_L^2}{2} J_c^2 = \frac{\mu_0 H_c^2}{2} \quad (68)$$

Therefore

$$\boxed{J_c^{\text{London}} = \frac{H_c}{\lambda_L}} \quad (69)$$

If the critical field is quoted through the induction $B_c = \mu_0 H_c$, this may equivalently be written as

$$\boxed{J_c^{\text{London}} = \frac{B_c}{\mu_0 \lambda_L}} \quad (70)$$

2.6 Cylindrical wire and the Silsbee criterion

Consider an infinitely long cylindrical superconducting wire of radius R , with

$$R \gg \lambda_L. \quad (71)$$

For a current I flowing along the wire, Ampère's law gives, at the surface $r = R$,

$$H(R) = \frac{I}{2\pi R}. \quad (72)$$

According to the Silsbee criterion, superconductivity is destroyed when the self-field at the surface reaches the thermodynamic critical field, $H(R) = H_c$.

Therefore

$$\boxed{I_c = 2\pi R H_c}. \quad (73)$$

Equivalently, using $B_c = \mu_0 H_c$,

$$\boxed{I_c = \frac{2\pi R B_c}{\mu_0}} \quad (74)$$

If we define the *average* critical current density over the full cross section of the wire,

$$\bar{J}_c = \frac{I_c}{\pi R^2}, \quad (75)$$

then

$$\bar{J}_c = \frac{2H_c}{R}. \quad (76)$$

This does not coincide with the London result

$$J_c^{\text{London}} = \frac{H_c}{\lambda_L}. \quad (77)$$

The reason is that for $R \gg \lambda_L$, the current is not uniformly distributed over the whole cross section. It is concentrated in a surface layer of thickness of order λ_L .

The effective area carrying the current is therefore approximately $A_{\text{eff}} \simeq 2\pi R\lambda_L$ and the corresponding local current density near the surface is then

$$J_{c,\text{surf}} \simeq \frac{I_c}{2\pi R\lambda_L} = \frac{H_c}{\lambda_L}. \quad (78)$$

Thus, we see that we recover the London result $J_{c,\text{surf}} = J_c^{\text{London}}$.

3. Critical current of a thin wire or film

In this section, the sample is assumed to be thin enough that the amplitude of the order parameter does not vary appreciably across its thickness, $d \ll \xi(T)$, so that

$$\psi(\mathbf{r}) = |\psi|e^{i\theta(\mathbf{r})}, \quad (79)$$

with $|\psi|$ approximately constant in space.

However, unlike in the London limit, the current can now be strong enough such that the amplitude $|\psi|$ is no longer equal to its unperturbed value $|\psi_\infty|$.

$$f_s = f_{n0} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*} \left| \left(\frac{\hbar}{i} \nabla - e^* \mathbf{A} \right) \psi \right|^2 + \frac{B^2}{2\mu_0}. \quad (80)$$

Using

$$m^* \mathbf{v}_s = \hbar \nabla \theta - e^* \mathbf{A}, \quad (81)$$

and assuming $|\psi|$ is spatially constant,

$$\left(\frac{\hbar}{i} \nabla - e^* \mathbf{A} \right) \psi = m^* \mathbf{v}_s \psi. \quad (82)$$

The kinetic-energy contribution in the Ginzburg–Landau free-energy density is therefore

$$\frac{1}{2m^*} \left| \left(\frac{\hbar}{i} \nabla - e^* \mathbf{A} \right) \psi \right|^2 = \frac{1}{2} m^* v_s^2 |\psi|^2 \quad (83)$$

3.1 Optimal value of $|\psi|$ for a given v_s

For fixed superfluid velocity v_s , the terms depending on $|\psi|$ in the GL free energy are

$$f_s = f_{n0} + \left(\alpha + \frac{1}{2}m^*v_s^2 \right) |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{B^2}{2\mu_0}. \quad (84)$$

To find the optimal value of $|\psi|$, we need to minimize with respect to $|\psi|^2$

Let $x = |\psi|^2$, then

$$f_s = f_{n0} + \left(\alpha + \frac{1}{2}m^*v_s^2 \right) x + \frac{\beta}{2} x^2 + \frac{B^2}{2\mu_0}. \quad (85)$$

The equilibrium condition is

$$\frac{\partial f_s}{\partial x} = 0, \quad (86)$$

which gives $\alpha + \frac{1}{2}m^*v_s^2 + \beta|\psi|^2 = 0$ and

$$\boxed{|\psi|^2 = -\frac{1}{\beta} \left(\alpha + \frac{1}{2}m^*v_s^2 \right)} \quad (87)$$

Using the bulk value $|\psi_\infty|^2 = -\frac{\alpha}{\beta}$, we define the function

$$g = \frac{|\psi|}{|\psi_\infty|} \quad (88)$$

Then

$$g^2 = \frac{|\psi|^2}{|\psi_\infty|^2} = 1 - \frac{m^*v_s^2}{2|\alpha|}. \quad (89)$$

Equivalently,

$$v_s = \sqrt{\frac{2|\alpha|}{m^*}} \sqrt{1 - g^2} \quad (90)$$

This result shows that increasing the superfluid velocity suppresses the superconducting order parameter. The amplitude vanishes when

$$v_s = \sqrt{\frac{2|\alpha|}{m^*}}. \quad (91)$$

However, the current reaches its maximum before this point.

3.2 Current density as a function of f

The superconducting current density is

$$\mathbf{J}_s = e^* |\psi|^2 \mathbf{v}_s. \quad (92)$$

Since $e^* = -2e$, it is convenient to work with the magnitude,

$$J_s = |e^*| |\psi|^2 v_s. \quad (93)$$

Using

$$|\psi|^2 = |\psi_\infty|^2 g^2 \quad (94)$$

and

$$v_s = \sqrt{\frac{2|\alpha|}{m^*}} \sqrt{1 - g^2}, \quad (95)$$

we obtain

$$J_s = |e^*| |\psi_\infty|^2 \sqrt{\frac{2|\alpha|}{m^*}} g^2 \sqrt{1 - g^2} \quad (96)$$

We define $J_0 = |e^*| |\psi_\infty|^2 \sqrt{\frac{2|\alpha|}{m^*}}$.

Then

$$\frac{J_s}{J_0} = g^2 \sqrt{1 - g^2}. \quad (97)$$

This function vanishes at both

$$g = 0 \quad (98)$$

and

$$g = 1. \quad (99)$$

At $g = 1$, the condensate has its unperturbed amplitude, corresponding to $v_s = 0$, and therefore the current is zero. As v_s increases, the current first increases, but the condensate is progressively suppressed. Eventually the decrease of $|\psi|^2$ dominates and the current decreases.

Maximum current and critical value of f

We maximize

$$\Delta(g) = g^2 \sqrt{1 - g^2}. \quad (100)$$

Its derivative is

$$\frac{d\Delta}{da} = 2g\sqrt{1 - g^2} - \frac{g^3}{\sqrt{1 - g^2}} \quad (101)$$

$$= \frac{g(2 - 3g^2)}{\sqrt{1 - g^2}}. \quad (102)$$

The nontrivial extremum is therefore given by

$$2 - 3g^2 = 0, \quad (103)$$

so that

$$g_c^2 = \frac{2}{3} \quad (104)$$

and

$$g_c = \sqrt{\frac{2}{3}}. \quad (105)$$

Thus, at the critical current,

$$|\psi_c|^2 = \frac{2}{3} |\psi_\infty|^2. \quad (106)$$

The corresponding critical superfluid velocity is obtained from

$$g_c^2 = 1 - \frac{m^* v_c^2}{2|\alpha|}. \quad (107)$$

which gives

$$v_c = \sqrt{\frac{2|\alpha|}{3m^*}}. \quad (108)$$

Critical current density

At the maximum,

$$J_c = J_0 \left(\frac{2}{3} \right) \sqrt{1 - \frac{2}{3}}. \quad (109)$$

Therefore

$$J_c = J_0 \frac{2}{3\sqrt{3}}. \quad (110)$$

Using the definition of J_0 ,

$$J_c^{\text{GL}} = \frac{2}{3\sqrt{3}} |e^*| |\psi_\infty|^2 \sqrt{\frac{2|\alpha|}{m^*}}. \quad (111)$$

We can express this in terms of H_c and λ_L .

$$J_c^{\text{GL}} = \frac{2\sqrt{2}}{3\sqrt{3}} \frac{H_c}{\lambda_L}. \quad (112)$$

Numerically,

$$\frac{2\sqrt{2}}{3\sqrt{3}} \simeq 0.544, \quad (113)$$

so

$$J_c^{\text{GL}} \simeq 0.544 \frac{H_c}{\lambda_L}. \quad (114)$$

Comparison with the London result

In the London approximation, we obtained

$$J_c^{\text{London}} = \frac{H_c}{\lambda_L}. \quad (115)$$

By contrast, GL theory gives

$$J_c^{\text{GL}} = \frac{2\sqrt{2}}{3\sqrt{3}} \frac{H_c}{\lambda_L}. \quad (116)$$

Therefore

$$J_c^{\text{GL}} < J_c^{\text{London}}. \quad (117)$$

The physical reason is that GL theory allows the current itself to suppress the superconducting condensate. As the superfluid velocity increases, $|\psi|$ decreases, and therefore the superfluid density also decreases.

The maximum current is reached before the order parameter vanishes completely.