

Superconductivity - M2

Kris Van Houcke, Claire Marrache-Kikuchi

TD 3: Type II superconductors and the vortex state

1 Domain-wall energy

To study the energy of a normal-superconducting domain wall normal at the critical magnetic field, we seek solutions of the GL differential equations subject to the boundary conditions

$$\begin{array}{llll} \psi(x) = 0 & \text{and} & B(x) = \mu_0 H_c & \text{as} & x \rightarrow -\infty, \\ \psi(x) = \psi_\infty & \text{and} & B(x) = 0 & \text{as} & x \rightarrow +\infty, \end{array}$$

with ψ_∞ the bulk value of the order parameter. Note that the problem is essentially one-dimensional. Since the two phases co-exist, we must be exactly on the transition curve (i.e. the external field H is fixed to $H_c(T)$) and the appropriate quantity to calculate is the Gibbs free energy density g . While H is fixed, B depends on the location of the domain wall.

1. The surface energy γ is the excess of Gibbs free energy G per unit area over what it would be if $B(x) = 0$ everywhere (bulk superconductor). Write down γ as an integral over the system involving ψ and B .
2. Use the GL differential equation to show that

$$\gamma = \int_{-\infty}^{+\infty} dx \left[-\frac{\beta}{2} |\psi|^4 + \frac{\mu_0}{2} \left(\frac{B(x)}{\mu_0} - H_c \right)^2 \right]. \quad (1)$$

3. Relate β to ψ_∞ and H_c , and show that

$$\gamma = \frac{\mu_0 H_c^2}{2} \int_{-\infty}^{+\infty} dx \left[-\left| \frac{\psi(x)}{\psi_\infty} \right|^4 + \left(1 - \frac{B(x)}{\mu_0 H_c} \right)^2 \right]. \quad (2)$$

Give an interpretation to the two terms appearing here. Sketch $\psi(x)$ and $B(x)$ in the domain wall for the cases with $\kappa = \lambda(T)/\xi(T) \ll 1$ and $\kappa \gg 1$. Is γ positive or negative in these cases?

It can be shown that γ changes sign at $\kappa = 1/\sqrt{2}$. In a ground-breaking paper Abrikosov showed that superconductors with $\kappa > 1/\sqrt{2}$ behave radically differently. His paper created the study of so-called type II superconductivity. The negative surface energy causes normal (magnetic flux-bearing) regions to subdivide until a quantum limit is reached in which each quantum of flux Φ_0 passes through the sample as a distinct flux tube¹.

¹Note that in the intermediate state of type I superconductors there are also normal flux-bearing regions. However, these regions are macroscopic, in contrast to the vortices in a type II superconductor, which contain only a single quantum of flux.

2 Linearized GL equations: nucleation of superconductivity

To avoid facing the full complexity of solving the pair of coupled nonlinear partial differential GL equations, one uses the *linearised* equation, which corresponds to dropping $\frac{1}{2}\beta|\psi|^4$ in the GL free energy density. The linearised first GL equation is

$$\left(\frac{\nabla}{i} - \frac{e_*\mathbf{A}}{\hbar}\right)^2 \psi = -\frac{2m_*\alpha}{\hbar^2}\psi = \frac{\psi}{\xi^2(T)}. \quad (3)$$

Note that this equation is identical to the Schrödinger equation for a free particle of mass m_* and charge e_* in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, with $|\alpha|$ playing the role of the energy eigenvalue. Moreover, we also take $\mathbf{A} = \mathbf{A}_{ext}$ since all screening effects due to supercurrents are proportional to $|\psi|^2$ (and are thus dropped in the linearised approximation).

1. Give a condition for $|\psi|^2$ compared to $|\psi_\infty|^2$ for this to be justified.

If solutions to the linearized GL equations exist for certain magnetic fields, then this will indicate that solutions to the full nonlinear GL equations are possible with infinitesimal amplitude $|\psi|$. The magnetic field values determined in this way determines (a) the critical field if the phase transition is continuous, (b) the nucleation field if the transition is of first order.

Let us first solve the problem of the nucleation of superconductivity in a bulk sample in presence of a field $\mathbf{H}$ along the z -axis. A convenient gauge choice is $\mathbf{A} = Bx \mathbf{e}_y$ (so-called Landau gauge).

2. Since the potential only depends on x , it is reasonable to look for a solution of the form

$$\psi = e^{ik_y y} e^{ik_z z} f(x). \quad (4)$$

Show that the linearised GL equation becomes

$$-f''(x) + \left(\frac{e_* B}{\hbar}\right)^2 (x - x_0)^2 f = \left(\frac{1}{\xi^2} - k_z^2\right) f. \quad (5)$$

Determine x_0 .

3. What does this equation describe? Give the spectrum. These levels are the so-called Landau levels.
4. Give the field H in terms of k_z , ξ , Φ_0 and n (which labels the Landau levels). Give the maximal value of H , to which we will refer as H_{c2} .
5. Verify that the corresponding eigenfunction is

$$f(x) = \exp\left[-\frac{(x - x_0)^2}{2\xi^2}\right]. \quad (6)$$

Note that ξ gives the typical length scale over which ψ varies.

6. Give a relation between H_{c2} and the thermodynamic critical field H_c . Which of these two is the biggest? Give a physical interpretation.

3 Bonus exercise: Structure and energy of an isolated vortex

We have found that superconductors with $\kappa > 1/\sqrt{2}$ have solutions of the GL equations with $|\psi| > 0$ until fields $H_{c2} > H_c$ are reached. Below H_{c2} there exists a so-called Abrikosov solution (also called Shubnikov state). This solution corresponds to a regular array of vortices of current surrounding nodal lines of ψ (see Fig. 4 for an experimental image of the Abrikosov vortex lattice). Each unit cell of the array carries a total flux equal to Φ_0 . We wish to find H_{c1} , the field at which flux first penetrates (see phase diagram in Fig. 2). When $H_{ext} = H_{c1}$ the Gibbs free energy must have the same value whether the first vortex is in or out the sample:

$$G_s|_{\text{no flux}} = G_s|_{\text{first vortex}} . \quad (7)$$

1. Start from Eq. (7) and show that

$$H_{c1} = \frac{\epsilon_1}{\Phi_0} , \quad (8)$$

with ϵ_1 the free energy per unit of length of the vortex line.

We wish to study the structure of a single vortex. It is convenient to introduce a vortex wave function of the form

$$\psi(\mathbf{r}) \equiv \psi(r, \theta) = |\psi_\infty| f(r) e^{i\theta} \quad (9)$$

which builds in the axial symmetry and the fact that the phase of ψ varies by 2π in making a complete circuit around the vortex core, corresponding to the existence of a single flux quantum going through the vortex. With the external magnetic field $\mathbf{B} = B(r)\mathbf{e}_z$ in the z -direction, we have for the vector potential $\mathbf{A} = A(r)\mathbf{e}_\theta$ (symmetric gauge).

2. Show that one has

$$A(r) \approx \frac{B(0)r}{2} \quad (10)$$

near the center of the vortex where the magnetic field is maximal and

$$A(r) = \frac{\Phi_0}{2\pi r} \quad (11)$$

far away from the vortex core.

3. Show that the GL equations become

$$f - f^3 - \xi^2 \left(\left(\frac{1}{r} - \frac{2\pi A}{\Phi_0} \right)^2 f - \frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) \right) = 0 \quad (12)$$

$$\mathbf{J} = \frac{e_* \hbar}{m_*} |\psi_\infty|^2 f^2 \left(\frac{1}{r} - \frac{2\pi A(r)}{\Phi_0} \right) \mathbf{e}_\theta . \quad (13)$$

The problem is now to find simultaneous solutions of these two non-linear differential equations for $f(r)$ and $A(r)$.

4. Since this requires numerical methods, let us examine the limiting cases. Let us first look at $r \rightarrow 0$. Let us assume that the solution starts as

$$f = cr^n \quad n \geq 1 . \quad (14)$$

Show that $n = 1$ and the first two leading contributions are

$$f(r) = cr \left(1 - \frac{r^2}{8\xi^2} \left(1 + \frac{B(0)}{\mu_0 H_{c2}} \right) \right) . \quad (15)$$

From higher order contributions one can get $c \sim 1/(2\xi)$. When does f start to saturate? One can show that the function $f(r) = \tanh(ar/\xi)$ with $a \sim 1$ is a reasonable approximation of the entire range (see Fig. 1).

5. There is a current circling the vortex core. If $\lambda \gg \xi$, show from Eq. (13) that London's equation is modified in the presence of a vortex line, due to the singular behaviour of the velocity field at the vortex core (see Eq. (13)):

$$\mu_0 \lambda_L^2 \nabla \times \mathbf{J} + \mathbf{B} = \Phi_0 \delta^{(2)}(\mathbf{r}) \mathbf{e}_z , \quad (16)$$

with $\delta^{(2)}(\mathbf{r})$ is the two-dimensional Dirac- δ function. The term on the right-hand-side ensures that there is a quantum of flux going through the core.

6. Start from Eq. (16) and obtain the magnetic field $B(r)$ for $r \gg \lambda$ and for $\xi \lesssim r \ll \lambda$.²
7. Now let us find ϵ_1 . Neglecting the core of radius ξ , we have (see TD 2, the free energy in the London approximation)

$$\epsilon_1 = \frac{1}{2\mu_0} \iint_{r>\xi} dS (\mathbf{B}^2 + \lambda_L^2 (\nabla \times \mathbf{B})^2) , \quad (17)$$

which corresponds to the magnetic energy and the kinetic energy of the currents. Use the vector identity $\nabla \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{D} \cdot (\nabla \times \mathbf{C}) - \mathbf{C} \cdot (\nabla \times \mathbf{D})$ and the modified London equation (16) to show

$$\epsilon_1 = \frac{\Phi_0 B(\xi)}{2\mu_0} . \quad (18)$$

8. Get an expression for H_{c1} in terms of the penetration and coherence lengths. Determine the ratio H_{c1}/H_c in terms of κ .

²The equation $\nabla^2 B(r) - \alpha^2 B(r) = -\beta \delta(\mathbf{r})$ has an exact solution $B(r) = \beta/2\pi K_0(\alpha r)$ where K_0 is the modified Bessel function of the second kind. Asymptotic expansions of K_0 give: $K_0(x) \simeq \sqrt{\pi/(2x)} e^{-x}$ for $x \gg 1$ and $K_0(x) \simeq \ln(1/x) + 0.12$ for $x \ll 1$.

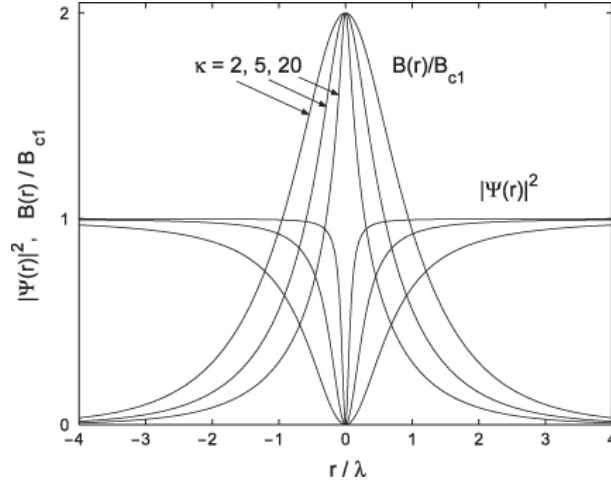


Figure 1: The magnetic field $B(r)$ and order parameter $|\psi(r)|^2$ of an isolated flux line calculated from Ginzburg-Landau theory for $\kappa = 2, 5$ and 20 . (Figure taken from E. H. Brandt, Supercond. Sci. Technol. 22, 034019 (2009))

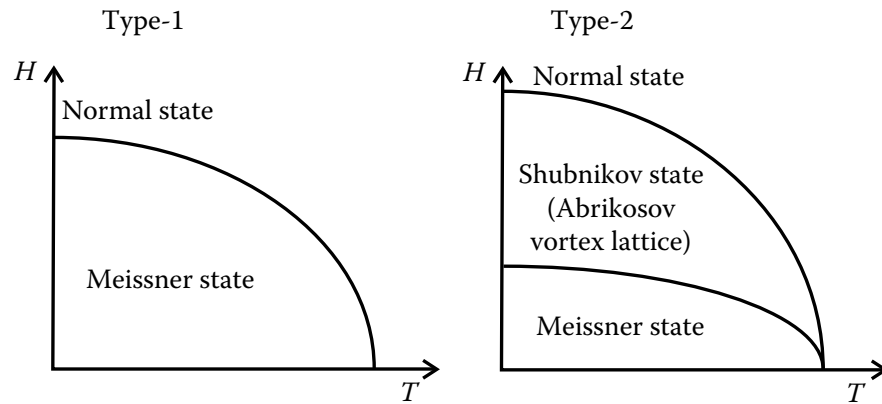


Figure 2: Schematic mean-field phase diagram of type I and type II superconductors. (Figure taken from *Superfluid states of matter*, B. Svistunov, E. Babaev, N. Prokof'ev.)

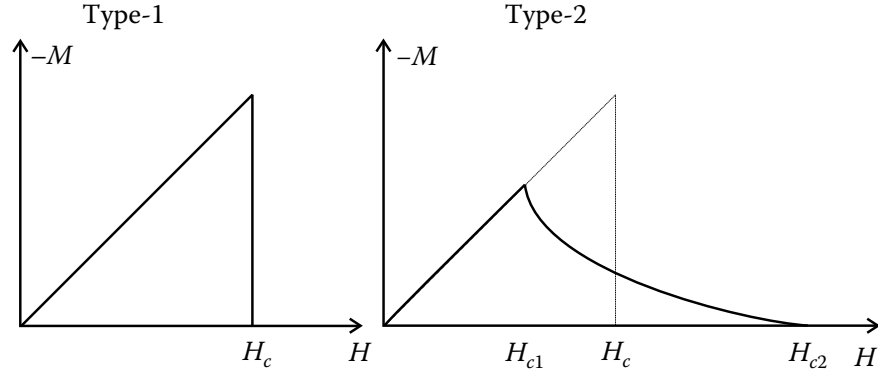


Figure 3: Sketch of magnetisation curves in a superconductor. The magnetic moment of the type-I superconductor is defined by perfect diamagnetic response in all fields below H_c . At $H = H_c$, the type-I superconductor undergoes the first-order phase transition to the normal state. In the type-II superconductor, the magnetisation undergoes a gradual change in fields $H_{c1} < H < H_{c2}$, due to the formation of the Abrikosov vortex lattice.

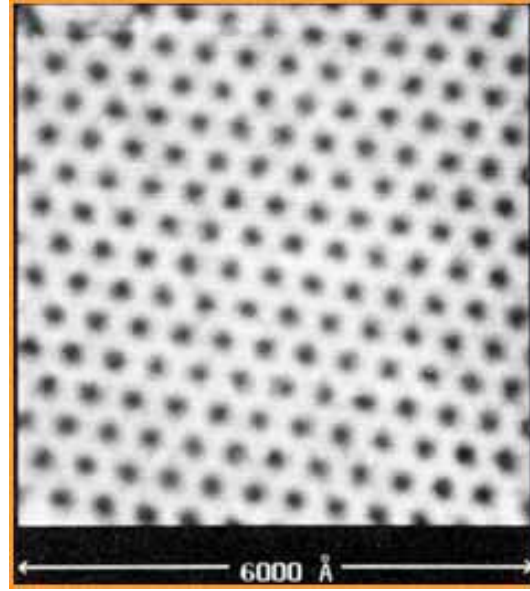


Figure 4: Abrikosov flux lattice produced by a 1 T magnetic field in NbSe_2 at 1.8 K. Figure taken from H.F. Hess et al., Phys. Rev. Lett. 62, 214 (1989).