

TD 1 - Landau theory.

1. Functional Derivation.

$$\vec{r} \in \mathbb{R}^d$$

$$\phi(\vec{r}) \text{ fct.}$$

$$\text{functional: } F[\phi(\vec{r})] \quad \phi(\vec{r}) \xrightarrow{\text{fct}} F[\phi] \in \mathbb{R} \text{ or } \mathbb{C}.$$

ex $F[\vec{B}] = \text{free energy for } \vec{B}(\vec{r})$. has to be min.

$$\frac{\delta \phi(r')}{\delta \phi(r)} = \delta(r-r')$$

$$\textcircled{1} \quad \frac{\delta}{\delta \phi(\vec{r})} \int d\rho (\phi(\rho))^u = \int d\rho u (\phi(\rho))^{u-1} \frac{\delta \phi(\rho)}{\delta \phi(r)} = \underline{u \phi(r)^{u-1}}$$

$$\textcircled{2} \quad \frac{\delta}{\delta \phi(\vec{r})} \left[\int d\rho \phi(\rho) \right]^u = \underline{u \left[\int d\rho \phi(\rho) \right]^{u-1}}$$

$$\textcircled{3} \quad \frac{\delta}{\delta \phi(\vec{r})} \int d\rho e^{\lambda \phi(\rho)} = \int d\rho \lambda e^{\lambda \phi(\rho)} \frac{\delta \phi(\rho)}{\delta \phi(r)} = \underline{\lambda e^{\lambda \phi(r)}}$$

$$\textcircled{4} \quad \frac{\delta}{\delta \phi(\vec{r})} \exp \left[\int d\rho h(\rho) \phi(\rho) \right] = \exp \left[\int \dots \right] \int d\rho h(\rho) \frac{\delta \phi(\rho)}{\delta \phi(r)} \\ = h(r) \exp \left[\int d\rho h(\rho) \phi(\rho) \right].$$

$$\textcircled{5} \quad \frac{\delta}{\delta \phi(\vec{r})} \int_a^b d\rho (\nabla \phi(\rho))^2 = \int_a^b d\rho 2 \nabla \phi \frac{\delta (\nabla \phi(\rho))}{\delta \phi(r)} = \int_a^b d\rho \nabla \phi \nabla \left(\frac{\delta \phi(\rho)}{\delta \phi(r)} \right)$$

∇ and $\frac{\delta}{\delta \phi}$ are 2

$$= \left[2 \nabla \phi \frac{\delta \phi(\rho)}{\delta \phi(r)} \right]_a^b - \int_a^b d\rho \Delta \phi \frac{\delta \phi(\rho)}{\delta \phi(r)} \quad \text{linear op.} \Rightarrow \text{can be interchanged.}$$

(integrating by part)

$$= \underline{2 \left\{ \phi'(b) \delta(r-b) - \phi'(a) \delta(r-a) \right\} - 2 \Delta \phi.}$$

boundary terms which can be neglected ($- \infty \leq \infty$) when whole domain. (if no domains)

$$\begin{aligned}
 \textcircled{6} \quad \frac{\delta}{\delta \phi(r)} \int d\rho (\Delta \phi)^2 &= 2 \int d\rho \underbrace{\Delta \phi}_\mu \underbrace{\Delta(\delta(r-\rho))}_{\nu'} \\
 &\stackrel{\text{IPP}}{=} 2 [\text{Bound.}] - 2 \int d\rho \underbrace{\phi'''}_\mu \underbrace{\nabla(\delta(r-\rho))}_{\nu'} \\
 &\stackrel{\text{IPP}}{=} -2 [\text{Bound.}] + \underline{2 \phi''''(r)}.
 \end{aligned}$$

$$\textcircled{7} \quad \frac{\delta}{\delta \vec{B}} \int d\rho (\vec{\nabla} \times \vec{B})^2 \quad (\vec{\nabla} \times \vec{B})_i = \epsilon_{ijk} \partial_j B_k. \quad (\text{Einstein notation}).$$

$$\int d\rho (\vec{\nabla} \times \vec{B})^2 = \int d\rho \underbrace{\epsilon_{ijk} \partial_j B_k}_\mu \underbrace{\epsilon_{imn} \partial_m B_n}_{\nu'}$$

$$\stackrel{\text{IPP}}{=} - \int d\rho \epsilon_{ijk} B_k \partial_j (\underbrace{\epsilon_{imn} \partial_m B_n}_{(\vec{\nabla} \times \vec{B})_i}). \quad \text{neglecting bound. terms.}$$

$$= + \int d\rho \vec{B} \cdot (\vec{\nabla} \times (\vec{\nabla} \times \vec{B})).$$

$$\frac{\delta}{\delta \vec{B}} \int d\rho (\vec{\nabla} \times \vec{B})^2 = \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) + \int d\rho \underbrace{\vec{B}}_\mu \cdot \underbrace{(\vec{\nabla} \times (\vec{\nabla} \times \frac{\delta \vec{B}(\rho)}{\delta \vec{B}(r)}))}_{\nu'}.$$

$$\stackrel{\text{IPP}}{=} \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) - \int d\rho \underbrace{(\vec{\nabla} \times \vec{B})}_\mu \cdot \underbrace{(\vec{\nabla} \times \delta(\rho-r))}_{\nu'}$$

$$\stackrel{\text{IPP}}{=} \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) + \int (\vec{\nabla} \times (\vec{\nabla} \times \vec{B})) \delta(r-\rho)$$

$$= \underline{2 \vec{\nabla} \times (\vec{\nabla} \times \vec{B})}.$$

2. London approximation of free energy.

• Def. of current density: $\vec{J}_s = -en_s \vec{v}_s$.

• 1st London eq: $\frac{\partial \vec{J}_s}{\partial t} = \frac{e^2 n_s}{m} \vec{E}$.

• Free energy: $F = E_0 + E_{kin} + E_{mag}$

$\underbrace{\int d^3\vec{r} \frac{1}{2\mu_0} B^2}_{\text{free enrg of condensed e- at rest}}$

$\underbrace{\int d^3\vec{r} \frac{1}{2} m n_s v_s^2(\vec{r})}_{\text{(ok if } \vec{v}_s(\vec{r}) \text{ varying slowly as fct of } \vec{r})}$

⊗ Cheating version
since we shld remove
enrg of mag. field but
est that does not change
final result

1) Maxwell-Ampere: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_s = -\mu_0 e n_s \vec{v}_s$.

$E_{kin} = \int d^3\vec{r} \frac{1}{2} m n_s \frac{1}{\mu_0^2 e^2 n_s} (\vec{\nabla} \times \vec{B})^2$

$F = E_0 + \frac{1}{2\mu_0} \int d^3\vec{r} \left\{ B^2 + \underbrace{\frac{m}{\mu_0 e^2 n_s}}_{\lambda_L^2} (\vec{\nabla} \times \vec{B})^2 \right\}$

$\lambda_L = \sqrt{\frac{m}{\mu_0 e^2 n_s}}$ London penetration depth.

2) $\frac{\delta F}{\delta \vec{B}} = 0 \Rightarrow$ minimizatiⁿ of F. wrt distributi^on of field.

$\frac{\delta F}{\delta \vec{B}} = \frac{1}{2\mu_0} \int d^3\vec{r} 2 \vec{B}(\vec{r}) \delta(\vec{r}-\vec{r}) + \frac{\lambda_L^2}{2\mu_0} \frac{\delta}{\delta \vec{B}} \int d^3\vec{r} (\vec{\nabla} \times \vec{B})^2$

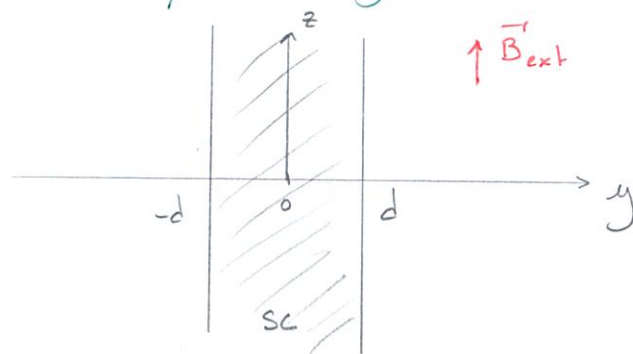
$= \frac{1}{\mu_0} \vec{B}(\vec{r}) + \frac{\lambda_L^2}{\mu_0} \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{0}$

$\vec{B}(\vec{r}) + \lambda_L^2 \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{0}$

eq. for the propagation
of $\vec{B}$.

$\vec{\nabla} \times (\lambda_L \vec{J}_s) + \vec{B} = \vec{0} = 2^{nd} \text{ London eq.}$

3. Superconducting slab.



$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0.$$

$$\vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{B}) = \Delta \vec{B}$$

$$\Delta \vec{B} = \frac{1}{\lambda_L^2} \vec{B}.$$

$\Rightarrow$ Messner effect (Messner + Ochsenfeld '33).

Equil. state where $E_{kin} + E_{mag}$ is min.
For macroscopic samples, this $\equiv$ to expulsion of magnetic flux.

2) Outside region: $\vec{B} = B_{ext} \vec{u}_z$.

Inside region: $\vec{B}(y) = (\alpha e^{-y/\lambda_L} + \beta e^{y/\lambda_L}) \vec{u}_z$.

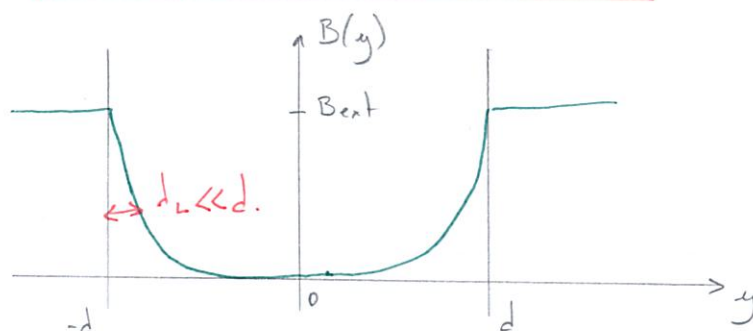
Continuity at $y = \pm d$: conservation of $\vec{B}_n$ at interface $\Rightarrow \vec{B}_n = 0$ (var PGdG).

$$B_{ext} = \alpha e^{-d/\lambda_L} + \beta e^{+d/\lambda_L} = \alpha e^{+d/\lambda_L} + \beta e^{-d/\lambda_L}.$$

$$\alpha (e^{+d/\lambda_L} - e^{-d/\lambda_L}) = \beta (e^{+d/\lambda_L} - e^{-d/\lambda_L}) \Rightarrow \alpha = \beta.$$

$$B_{ext} = 2\alpha \cosh(d/\lambda_L).$$

$$\vec{B}(y) = B_{ext} \frac{\cosh(y/\lambda_L)}{\cosh(d/\lambda_L)} \vec{u}_z.$$



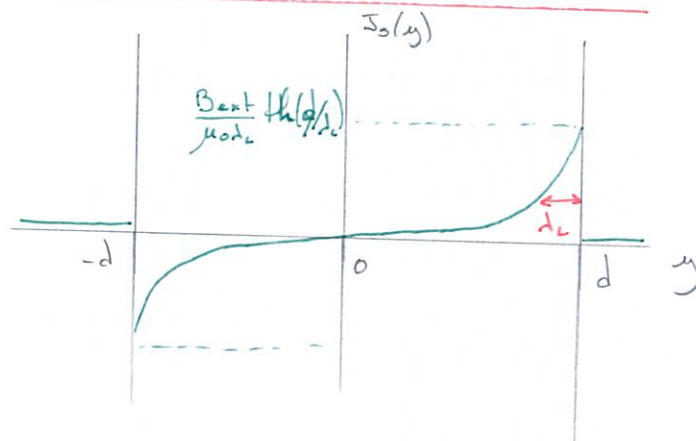
Messner effect.

$$3) \vec{B} \parallel \vec{u}_z \Rightarrow \vec{J}_s \parallel \vec{u}_x.$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_s = \vec{\nabla} \times (B(y) \vec{u}_z) = \frac{\partial B(y)}{\partial y} \vec{u}_x.$$

$$\hookrightarrow \vec{J}_s = J_s(y) \vec{u}_x$$

$$J_s(y) = \frac{B_{\text{ext}}}{\mu_0 \lambda_L} \frac{\sinh(y/\lambda_L)}{\cosh(d/\lambda_L)}.$$



Screening current are at the surf of the slab.

$$4) \vec{\eta} = \frac{\vec{B}}{\mu_0} - \frac{\vec{B}_{\text{ext}}}{\mu_0} = \eta(y) \vec{u}_z.$$

$$\vec{\eta} = \frac{1}{2d} \int_{-d}^d dy \frac{B_{\text{ext}}}{\mu_0} \left(\frac{\cosh(y/\lambda_L)}{\cosh(d/\lambda_L)} - 1 \right)$$

$$= \frac{2\lambda_L}{2d\mu_0} B_{\text{ext}} \th(\frac{d}{\lambda_L}) - \frac{B_{\text{ext}}}{\mu_0}$$

$$\vec{\eta} = -\frac{B_{\text{ext}}}{\mu_0} \left\{ 1 - \frac{\lambda_L}{d} \th(\frac{d}{\lambda_L}) \right\}.$$

Susceptibility: $\chi = \frac{\partial \vec{\eta}}{\partial H_{\text{ext}}} \Big|_{H_{\text{ext}}=0}$

$$\chi = - \left\{ 1 - \frac{\lambda_L}{d} \th(\frac{d}{\lambda_L}) \right\}. \quad d \gg \lambda_L \quad \chi \rightarrow -1 \quad \text{perfect diamagnet}$$

$$\th x \sim x - \frac{x^3}{3}$$

$$d \ll \lambda_L$$

$$\chi = - \left\{ 1 - \frac{\lambda_L}{d} \left(\frac{d}{\lambda_L} - \frac{1}{3} \frac{d^3}{\lambda_L^3} \right) \right\}.$$

$$\chi = \frac{1}{3} \frac{d^3}{\lambda_L^3}$$

4. Relevant lengthscales in a SC.

• Coherence length.

* Only e- $\sim k_B T_c$ of E_F play a role. $\Rightarrow \delta p \sim \frac{k_B T_c}{v_F}$

* Uncertainty principle: $\delta x \gtrsim \frac{\hbar}{\delta p} \sim \frac{\hbar v_F}{k_B T_c}$

* $\xi_0 = \frac{\hbar v_F}{\pi \Delta} \leftarrow \text{gap.}$

1) λ_L defined at $T=0$: $\mu_s = \mu = \frac{m}{\mu_0 e^2 \lambda_L^2}$

For Al: $\mu = 2700 \frac{\text{kg}}{\text{m}^3} \frac{6.02 \cdot 10^{23} \text{ at/mol}}{0.027 \text{ kg/mol}} \sim 6.02 \cdot 10^{29} \text{ m}^{-3}$

Scale over which the fields vary. $\lambda_L = \sqrt{\frac{m}{\mu_0 e^2 \mu_s}} = \sqrt{\frac{9 \cdot 10^{-31} \text{ kg}}{4 \pi \cdot 10^{-7} \cdot 1.05 \cdot 10^{34}}} = 2.2 \cdot 10^{-8} = 22 \text{ nm} \gg \text{\AA}$

$\xi_0 = \frac{\hbar v_F}{\Delta \bar{u}} = \frac{\hbar v_F}{1.76 k_B T_c}$

$E_F = \frac{\hbar^2 k_F^2}{2m}$

and $k_F = (3\pi^2 \mu)^{1/3} = 1.2 \cdot 10^{10} \text{ m}^{-1}$

$\frac{\hbar k_F}{m} = v_F = 1.2 \cdot 10^6 \text{ m/s}$

$\xi_0 = 1.3 \cdot 10^{-6} \text{ m} = 1.3 \mu\text{m} = 1300 \text{ nm} \gg \text{\AA}$

↑ scale over which e- are correlated in a Cooper pair

"Slowly varying" means $\lambda_L \gg \xi_0$, which is not the case here.

3) $\lambda_L \sim 2000 \text{\AA}$
 $T_c \sim 18 \text{ K}$
 $\xi_0 \sim 50 \text{\AA}$ } $\Rightarrow$ London theory ok.