

Superconductivity - M2 ICFP

Kris Van Houcke, Claire Marrache-Kikuchi

TD 1: London theory

The London equations, developed by brothers Fritz and Heinz London in 1935, give a simple phenomenological description of superconductors. The two equations describe the perfect conductivity and the Meissner effect.

1 Preliminaries: functional derivation

Let us consider a field $\phi(r)$ (here $r \in \mathbb{R}^d$ but I don't bother with the vector notation). The fundamental relation that allows the calculation of functional derivatives is:

$$\frac{\delta\phi(r')}{\delta\phi(r)} = \delta(r - r') . \quad (1)$$

Calculate

$$\begin{aligned} & \frac{\delta}{\delta\phi(r)} \int d\rho \phi(\rho)^n , \quad \frac{\delta}{\delta\phi(r)} \left[\int d\rho \phi(\rho) \right]^n , \quad \frac{\delta}{\delta\phi(r)} \left[\int d\rho \phi(\rho)^n \right]^m , \quad \frac{\delta}{\delta\phi(r)} \int d\rho e^{\lambda\phi(\rho)} \\ & \frac{\delta}{\delta\phi(r)} e^{\int d\rho h(\rho)\phi(\rho)} , \quad \frac{\delta}{\delta\phi(r)} \int d\rho (\nabla\phi)^2 , \quad \frac{\delta}{\delta\phi(r)} \int d\rho (\Delta\phi)^2 , \quad \frac{\delta}{\delta\mathbf{B}} \int d\rho (\nabla \times \mathbf{B})^2 , \dots \end{aligned}$$

(you are encouraged to invent other examples...)

2 The London approximation for the free energy

We consider the situation where there are supercurrents $\mathbf{J}_s(\mathbf{r})$ and associated magnetic fields $\mathbf{B}(\mathbf{r})$ in a sample in the superconducting state. Let $\mathbf{v}_s(\mathbf{r})$ be the drift (or superconducting) velocity of the electrons at point $\mathbf{r}$. It is related to the current density $\mathbf{J}_s$ by

$$\mathbf{J}_s(\mathbf{r}) = -e n_s \mathbf{v}_s(\mathbf{r}) , \quad (2)$$

where $-e$ is the electron charge and n_s the density of superconducting electrons¹.

The first London equation, $\frac{\partial \mathbf{J}_s}{\partial t} = \frac{e^2 n_s}{m} \mathbf{E}$, describes perfect conductivity since any electric field $\mathbf{E}$ *accelerates* the superconducting electrons rather than simply sustaining their velocity against resistance as described by Ohm's law for a normal conductor.

The second London equation can be obtained by considering the free energy of a superconductor as just the sum of the kinetic energy and the magnetic energy:

$$F = \mathcal{E}_0 + \mathcal{E}_{kin} + \mathcal{E}_{mag} , \quad (3)$$

¹Within London theory, the superfluid density n_s is treated as given. At this point, there is no way to understand its microscopic origin, or the dependence of n_s on temperature or applied magnetic field.

and with $\mathcal{E}_0$ the energy of the electrons in the superconducting state at rest. The kinetic is written as

$$\mathcal{E}_{kin} = \int d\mathbf{r} \frac{1}{2} m n_s v_s^2(\mathbf{r}) , \quad (4)$$

with m the effective mass of the electrons (in the conduction band). Eq. (4) would be exact for situations of uniform flow ($v_s = \text{const}$). It remains approximately correct provided that $v_s(\mathbf{r})$ is a *slowly varying function of $\mathbf{r}$* . We therefore assume that this is the case and that all fields and currents are weak. We return to this limitation in Section 4. The magnetic energy is given by

$$\mathcal{E}_{mag} = \int d\mathbf{r} \frac{1}{2\mu_0} (\nabla \times \mathbf{A})^2 , \quad (5)$$

with $\mathbf{A}$ the magnetic vector potential.

1. Show that one can express F as

$$F = \mathcal{E}_0 + \frac{1}{2\mu_0} \int d\mathbf{r} (\mathbf{B}^2 + \lambda_L^2 (\nabla \times \mathbf{B})^2) . \quad (6)$$

Give the expression for λ_L .

2. The equations governing variations of the magnetic field in space immediately follow from this. Take the functional derivative of F with respect to $\mathbf{B}$ and show that

$$\mathbf{B} + \lambda_L^2 \nabla \times (\nabla \times \mathbf{B}) = 0 . \quad (7)$$

This equation was first proposed by two brothers, Fritz and Heinz London, in 1935.

3 Superconducting slab

Consider an infinite superconducting slab bounded by two parallel planes perpendicular to the y -axis at $y = \pm d$. Let a uniform magnetic field of strength B_{ext} be applied along the z -axis.

1. Use the (so-called second) London equation to show that

$$\nabla^2 \mathbf{B} = \frac{\mathbf{B}}{\lambda_L^2} \quad (8)$$

in a superconductor.

2. Taking as a boundary condition that the parallel component of $\mathbf{B}$ be continuous at the surface. Deduce that within the superconductor

$$\begin{aligned} \mathbf{B} &= B(y) \hat{\mathbf{z}} , \\ B(y) &= B_{ext} \frac{\cosh(y/\lambda_L)}{\cosh(d/\lambda_L)} . \end{aligned} \quad (9)$$

Which effect is captured by this equation? Make a figure of the magnetic field as a function of y .

3. Show that the diamagnetic current density flowing in equilibrium is

$$\begin{aligned}\mathbf{J} &= J(y)\hat{\mathbf{x}}, \\ J(y) &= \frac{1}{\mu_0\lambda_L}B_{ext}\frac{\sinh(y/\lambda_L)}{\cosh(d/\lambda_L)}.\end{aligned}\tag{10}$$

Make a figure of the current density as a function of y .

4. The magnetisation density at a point within the slab is $\mathbf{M} = \mathbf{B}/\mu_0 - \mathbf{H}_{ext}$. Show that the average magnetisation density (averaged over the thickness of the slab) is

$$\overline{M} = -\frac{1}{\mu_0}B_{ext}\left(1 - \frac{\lambda_L}{d}\tanh(d/\lambda_L)\right),\tag{11}$$

and give the limiting form for the susceptibility when the slab is thick ($d \gg \lambda_L$) and thin ($d \ll \lambda_L$).

4 Relevant length scales in a superconductor

In this section we discuss the different length scales that determine the properties of a superconductor. It turns out that there are *three* such length scales. We have already encountered one: the *penetration depth*, which is given by λ_L (at $T = 0$) within the London approximation.

A second length, ξ_0 , is called the *coherence length*. Only electrons within $\sim k_B T_c$ of the Fermi energy can play a major role in a phenomenon which sets in at T_c , and these electrons have a momentum range $\delta p \approx k_B T_c / v_F$, where v_F is the Fermi velocity. Thus, by the uncertainty relation,

$$\delta x \gtrsim \hbar / \delta p \approx \hbar v_F / k_B T_c,\tag{12}$$

leading to the definition of a characteristic length

$$\xi_0 = a \frac{\hbar v_F}{k_B T_c},\tag{13}$$

where a is just some numerical constant of order unity. After the microscopic theory of superconductivity was completed by Bardeen, Cooper and Schrieffer (so-called BCS theory, which we will discuss later), it became clear that this length is closely related to the value of the energy gap Δ in the one-particle spectrum, which can be measured by various experiments². According to BCS theory³, one has $2\Delta(T=0) = 3.528 k_B T_c$. This leads one to take

$$\xi_0 = \frac{\hbar v_F}{\pi \Delta},\tag{14}$$

where the factor $1/\pi$ is arbitrary (but convenient in calculations). In other words, ξ_0 represents the size of a wave packet of superconducting charge carriers⁴.

²This energy gap is an experimental fact and will be an essential ingredient of the BCS theory. Note however that the existence of an energy gap is not a necessary condition for the existence of permanent current (superfluidity). A number of situations have been found where superfluidity occurs with no gap in the one-particle spectrum.

³We will explicitly show this in a later TD.

⁴Running ahead of things, the physical interpretation of ξ_0 is that it represents the physical size of a Cooper-pair bound state in the BCS theory.

1. Assume that at $T = 0$, n_s is equal to the density of conduction electrons, n . Calculate λ_L and ξ_0 explicitly for a typical elemental superconductor such as Al, where m is close to the free electron mass and $T_c = 1.2 \text{ K}$. Al has atomic number $A = 27$ and mass density $\rho = 2.7 \text{ g/cm}^3$.
2. The validity of the London equation required a slow variation in space of $\mathbf{v}_s(\mathbf{r})$ (and thus of the supercurrent $\mathbf{J}_s(\mathbf{r})$) and the field $\mathbf{B}(\mathbf{r})$. What is the typical length scale over which these vary? What do we mean by slow variation? Write down the validity condition in terms of the two length scales. Does the London theory apply for the case of Al in weak fields?
3. For transition metals and intermetallic compounds m is very large, so that λ_L is large ($\sim 2000 \text{ \AA}$). On the other hand, the transition temperature above which superconductivity disappears is high ($T_c = 18 \text{ K}$), leading to a small $\xi_0 \sim 50 \text{ \AA}$. Does the London equation apply?

In case the London equation does not apply, one can still observe a Meissner effect, but in order to calculate the penetration depth it is necessary to replace Eq. (7) by a more complicated relation, proposed by Pippard. These are called type-I superconductors or Pippard superconductors. In case the London theory for weak fields is applicable, one speaks of type-II superconductors or London superconductors. The distinction between the two classes is crucial for experiments made in the presence of external fields, as we will see later. Historically, in the first decades after the discovery of the Meissner effect, experiments were mainly done with type-I superconductors. Paradoxically, the theory was developed in the inverse order. The London brothers introduced their theory in 1935, but the necessary modifications for the type I superconductors was only proposed by Pippard in 1953.

In case of superconducting alloys, the coherence length and the penetration depth are modified by mean-free path effects. So, a third relevant length scale is the *mean free path* of the electrons at the Fermi surface of the metal,

$$\ell = v_F \tau, \quad (15)$$

with τ the scattering time from the Drude conductivity formula and v_F the electron band velocity at the Fermi surface. Qualitatively, if the mean free path is short due to disorder in the structure, the coherence length becomes smaller than ξ_0 and the penetration depth will be larger than λ_L . Therefore it frequently occurs that the addition of impurities into a Pippard superconductor transforms it into a London superconductor. If $\ell \gg \xi_0$ the superconductor is said to be in the *clean limit*, while if $\ell < \xi_0$ the superconductor is said to be in the *dirty limit*. It is a surprising and very important property of most superconductors that they can remain superconducting even when there are large numbers of impurities making the mean free path ℓ very short.