

# Superconductivity - M2

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## TD 2: Ginzburg-Landau theory

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In the Ginzburg-Landau (GL) theory the (Helmholtz) free energy density of the superconducting phase  $f_s$  is given as a functional of the order parameter  $\psi(\mathbf{r})$  and the magnetic field  $\mathbf{B} = \nabla \times \mathbf{A}$ :

$$f_s = f_{n0} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m_*} \left| \left( \frac{\hbar}{i} \nabla - e_* \mathbf{A} \right) \psi \right|^2 + \frac{\mathbf{B}^2}{2\mu_0}, \quad (1)$$

with  $f_{n0}$  the free energy density of the normal phase in absence of magnetic field. Here,  $e_* = -2e$ . The explanation for this charge only became clear after the development of the BCS theory<sup>1</sup>. Here,  $m_*$  plays the role of an effective mass for the quantum system with macroscopic wave function  $\psi(\mathbf{r})$ . From the GL functional one can derive the GL equations (see lecture)

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m_*} \left( \frac{\hbar}{i} \nabla - e_* \mathbf{A} \right)^2 \psi = 0, \quad (2)$$

$$\mathbf{J}_s = -i \frac{e_* \hbar}{2m_*} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{e_*^2}{m_*} |\psi|^2 \mathbf{A}. \quad (3)$$

A boundary condition, which assures that no current passes through the surface, is

$$\left( \frac{\hbar}{i} \nabla \psi - e_* \mathbf{A} \psi \right) \cdot \mathbf{n} = 0, \quad (4)$$

where  $\mathbf{n}$  is the unit vector normal to the superconductor's surface. After writing  $\psi(\mathbf{r}) = |\psi(\mathbf{r})| \exp(i\theta(\mathbf{r}))$ , one also has

$$m_* \mathbf{v}_s = \mathbf{p} - e_* \mathbf{A} = \hbar \nabla \theta - e_* \mathbf{A}, \quad (5)$$

with  $\mathbf{p}$  the generalised or dynamic momentum, associated with  $-i\hbar \nabla$ , and  $m_* \mathbf{v}_s$  the kinematic momentum.

## 1 Gauge invariance

The electrical potential  $V$  and the magnetic vector potential  $\mathbf{A}$  are not uniquely defined. Write down the gauge transformation that leaves the electrical field  $\mathbf{E}$  and the magnetic field  $\mathbf{B}$  unchanged. Show that the invariance of the GL equations under a local gauge transformation can only take place if the order parameter  $\psi$  is subjected to the appropriate phase transformation.

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<sup>1</sup>In fact, in their original paper GL assumed that the effective charge would be  $-e$ , not  $-2e$ . Ginzburg, then a young researcher, told his famous supervisor Landau that  $-2e$  fitted the available experimental data better than  $-e$ , but Landau overruled him and insisted that he was sure that it must be  $-e$ ! Ginzburg shared the 2003 Nobel prize in physics for his work in developing the GL model. (Footnote taken from the book *Superconductivity, Superfluids and Condensates* by J.F. Annet.)

## 2 The London limit

GL theory is a generalization of the London theory<sup>2</sup> to deal with situations in which  $n_s$  varies in space, and also to deal with the non-linear response to fields that are strong enough to change  $n_s$  (we will come back to this in section ??).

1. A bulk superconductor has a ground state with a constant order parameter  $\psi_\infty$ , a notation conventionally used because  $\psi$  approaches this value infinitely deep in the interior of the superconductor, where it is screened from any surface fields or currents. Consider the free-energy density in absence of fields and gradients. Give the solution  $|\psi_\infty|$ .
2. There must be a *phase-stiffness*, or an energy cost associated with changing the phase of  $\psi$  from one part of the sample to another. Assume  $|\psi|^2 = |\psi_\infty|^2$  constant, while the phase  $\theta(\mathbf{r})$  varies only slowly with position  $\mathbf{r}$ . Give the total free energy  $F_s$ , depending on the *superfluid stiffness*

$$\rho_s = \frac{\hbar^2}{2m_*} |\psi|^2. \quad (6)$$

3. Calculate the current density via the functional derivative of the free energy,

$$\mathbf{J}_s = -\frac{\delta F_s[\mathbf{A}]}{\delta \mathbf{A}}. \quad (7)$$

4. Retrieve the second London equation<sup>3</sup>. Give the expression for the penetration depth.
5. Use the definition of the thermodynamic critical field  $H_c$  to determine the critical current density in the London approximation.
6. Consider an infinitely long superconducting wire of cylindrical shape with radius  $R$  with  $R \gg \lambda_L$ . The Silsbee criterion states that the critical value of the current to destroy superconductivity is that at which the magnetic field due to the current itself is equal to the critical magnetic field. Give the critical current  $I_c$  as a function of the radius  $R$  and the critical magnetic field  $H_c$ . Give the critical current density  $J_c$ . Does it agree with the value obtained via London theory?

## 3 Critical current of a thin wire or film

We again consider the case when the amplitude  $|\psi(\mathbf{r})|$  is the same at all points. We have already encountered such a situation in section ??, where we had a bulk sample in a weak magnetic field, such that  $|\psi| = |\psi_\infty|$ . We now consider a very different situation: thin samples (films or wires) where  $|\psi|$  does not substantially vary within the thickness  $d \ll \xi(T)$  with  $\xi(T)$  the GL coherence length, but where the field  $\mathbf{B}$  (or the currents  $J_s$ ) are strong such that  $|\psi|$  is not necessarily equal to its unperturbed value  $|\psi_\infty|$ . One can thus approximate  $\psi(\mathbf{r})$  by  $|\psi|e^{i\theta(\mathbf{r})}$ , where  $|\psi|$  is constant.

<sup>2</sup>Note that, just like the London theory, GL theory makes the approximation of *local* electrodynamics. The non-locality becomes irrelevant when close to  $T_c$  since the penetration depth  $\lambda(T) \propto n_s^{-1/2} \propto (1 - T/T_c)^{-1/2} \gg \xi_0$  near  $T_c$ , with  $\xi_0$  Pippard's coherence length.

<sup>3</sup>The second London equation is applicable only when the following conditions are satisfied: (i)  $\lambda(T) \gg \xi_0 = 0.18\hbar v_F / k_B T_c$ , (ii)  $n_s(\mathbf{r})$  is nearly constant in space, and (iii)  $\mathbf{B}$  is small and can be treated as a perturbation.

1. For a given  $v_s$ , find the optimal value of  $|\psi|$ .
2. Give an expression for the corresponding current density  $J_s$  as function of  $f = |\psi|/|\psi_\infty|$ . Make a figure. What is the critical current density? Does the GL critical current density agree with that of the London theory? Why?