

(from J.F. Annett "Superconductivity, Superfluids and Condensates")

(we keep continuous numbering for the GL theory section)

4.3 Ginzburg–Landau theory of the bulk phase transition

The GL theory of superconductivity is built upon a general approach to the theory of second-order phase transitions which Landau had developed in the 1930s. Landau had noticed that typically second-order phase transitions, such as the Curie temperature in a ferromagnet, involve some change in symmetry of the system. For example, a magnet above the Curie temperature, T_c , has no magnetic moment. But below T_c a spontaneous magnetic moment develops. In principle this could point in any one of a number of different directions, each with an equal energy, but the system spontaneously chooses one particular direction. In Landau's theory such phase transitions are characterized by an

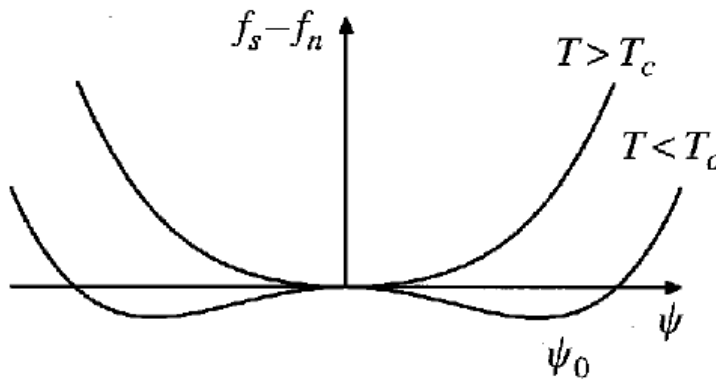


Fig. 4.2 Free energy difference between the normal and superconducting states (per unit volume) as a function of the order parameter ψ . For $T < T_c$ the free energy has a minimum at ψ_0 , while for $T > T_c$ the only minimum is at $\psi = 0$.

order parameter which is zero in the disordered state above T_c , but becomes nonzero below T_c . In the case of a magnet the magnetization, $\mathbf{M}(\mathbf{r})$, is a suitable order parameter.

For superconductivity Ginzburg and Landau (GL) postulated the existence of an order parameter denoted by ψ . This characterizes the superconducting state, in the same way as the magnetization does in a ferromagnet. The order parameter is assumed to be some (unspecified) physical quantity which characterizes the state of the system. In the normal metallic state above the critical temperature T_c of the superconductor it is zero. While in the superconducting state below T_c it is nonzero. Therefore it is assumed to obey:

$$\psi = \begin{cases} 0 & T > T_c, \\ \psi(T) \neq 0 & T < T_c. \end{cases} \quad (4.18)$$

GL postulated that the order parameter ψ should be a complex number, thinking of it as a macroscopic wave function for the superconductor in analogy with superfluid ^4He . At the time of their original work the physical significance of this complex ψ in superconductors was not at all clear. But, as we shall see below, in the microscopic BCS theory of superconductivity there appears a parameter, Δ , which is also complex. Gor'kov was able to derive the GL theory from BCS theory, and show that ψ is essentially the same as Δ , except for some constant numerical factors. In fact, we can even identify $|\psi|^2$ as the density of BCS "Cooper pairs" present in the sample.

GL assumed that the free energy of the superconductor must depend smoothly on the parameter ψ . Since ψ is complex and the free energy must be real, the energy can only depend on $|\psi|$. Furthermore, since ψ goes to zero at the critical temperature, T_c , we can Taylor expand the free energy in powers of $|\psi|$. For temperatures close to T_c only the first two terms in the expansion should be necessary, and so the free energy density ($f = F/V$) must be of the form:

$$f_s(T) = f_n(T) + a(T)|\psi|^2 + \frac{1}{2}b(T)|\psi|^4 + \dots \quad (4.19)$$

since $|\psi|$ is small. Here $f_s(T)$ and $f_n(T)$ are the superconducting state and normal state free energy densities, respectively. Clearly Eq. 4.19 is the only possible function which is real for any complex ψ near $\psi = 0$ and which is a differentiable function of ψ and ψ^* near to $\psi = 0$. The parameters $a(T)$ and $b(T)$ are, in general, temperature dependent phenomenological parameters of the theory. However, it is assumed that they must be smooth functions of temperature. We must also assume that $b(T)$ is positive, since otherwise the free energy density would have no minimum, which would be unphysical (or we would have to extend the expansion to include higher powers such as $|\psi|^6$).

Plotting $f_s - f_n$ as a function of ψ is easy to see that there are two possible curves, depending on the sign of the parameter $a(T)$, as shown in Fig. 4.2. In the case $a(T) > 0$, the curve has one minimum at $\psi = 0$. On the other hand, for $a(T) < 0$ there are minima wherever $|\psi|^2 = -a(T)/b(T)$. Landau and Ginzburg assumed that at high temperatures, above T_c , we have $a(T)$ positive, and hence the minimum free energy solution is one with $\psi = 0$, that is, the normal state. But if $a(T)$ gradually decreases as the temperature T is reduced, then the state of the system will change suddenly when we reach the point $a(T) = 0$. Below this temperature the minimum free energy solution changes to one with $\psi \neq 0$. Therefore we can identify the temperature where $a(T)$ becomes zero as the critical temperature T_c .

Near to this critical temperature, T_c , assuming that the coefficients $a(T)$ and $b(T)$ change smoothly with temperature, we can make a Taylor expansion,

$$\begin{aligned} a(T) &\approx \dot{a} \times (T - T_c) + \dots, \\ b(T) &\approx b + \dots, \end{aligned} \quad (4.20)$$

where $\dot{a}$ and b are two phenomenological constants. Then for temperatures just above T_c , $a(T)$ will be positive, and we have the free energy minimum, $\psi = 0$. On the other hand, just below T_c we will have minimum energy solutions with nonzero $|\psi|$, as seen in Fig. 4.2. In terms of the parameters $\dot{a}$ and b it is easy to see that

$$|\psi| = \begin{cases} \left(\frac{\dot{a}}{b}\right)^{1/2} (T_c - T)^{1/2} & T < T_c, \\ 0 & T > T_c. \end{cases} \quad (4.21)$$

The corresponding curve of $|\psi|$ as a function of temperature, T , is shown in Fig. 4.3. One can see the abrupt change from zero to nonzero values at the critical temperature T_c . In fact, this curve is qualitatively similar to those obtained with other types of second-order phase transitions within Landau's general theory. For example, the behavior of the order parameter ψ near T_c in Fig. 4.3 resembles closely change in the magnetization $\mathbf{M}$ in a ferromagnet near its Curie point in the Stoner theory of magnetism (Blundell 2001).

It turns out to be very important that, because ψ is complex, there are in fact an infinite set of minima corresponding to all possible values of the complex phase θ ,

$$\psi = |\psi|e^{i\theta}. \quad (4.22)$$

The phase value, θ is arbitrary, since all values lead to the same total free energy. But, just as in the case of the direction of magnetization $\mathbf{M}$ in a ferromagnet the system spontaneously chooses one particular value. A magnet heated to above T_c and then cooled again will almost certainly adopt a different random direction of magnetization, and the same would be true for the angle θ in a superconductor. In fact we have met this same concept before, in Chapter 2, when we discussed the XY symmetry of the macroscopic wave function in superfluid He II (Fig. 2.4).

The value of the minimum free energy in Fig. 4.2, is easily found to be $-a(T)^2/2b(T)$. This is the free energy difference (per unit volume) between the superconducting and non-superconducting phases of the system at temperature T . This corresponds to the condensation energy of the superconductor, and so we can write

$$f_s(T) - f_n(T) = -\frac{\dot{a}^2 (T - T_c)^2}{2b} = -\mu_0 \frac{H_c^2}{2}, \quad (4.23)$$

giving the thermodynamic critical field,

$$H_c = \frac{\dot{a}}{(\mu_0 b)^{1/2}} (T_c - T) \quad (4.24)$$

near to T_c .

From this free energy we can also obtain other relevant physical quantities, such as the entropy and heat capacity. Differentiating f with respect to T gives

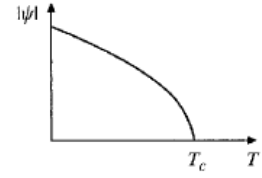


Fig. 4.3 Order parameter magnitude, $|\psi|$, as a function of temperature in the GL model.

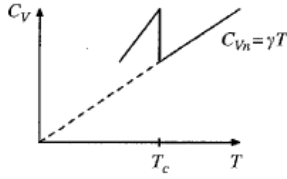


Fig. 4.4 Specific heat of a superconductor near T_c in the GL model. Above T_c the specific heat is given by the Sommerfeld theory of metals, $C_{Vn} = \gamma T$. At T_c there is a discontinuity and a change of slope.

³The GL theory can only be reliably used at temperatures close to T_c . Therefore our calculated specific heat is only correct near to T_c , and we cannot legitimately continue the GL line in Fig. 4.4 down from T_c all the way to $T = 0$.

⁴In fact the differences are deceptive! Our theory rests on a mean-field approximation and has neglected important thermal fluctuation effects, as we shall see below. When these fluctuations are large, as in the case of high temperature superconductors, the observed specific heat near T_c appears to show exactly the same XY universality class as the lambda point in superfluid helium (Overend, Howson and Lawrie 1994).

the entropy per unit volume, $s = S/V$,

$$s_s(T) - s_n(T) = -\frac{\dot{a}^2}{b}(T_c - T), \quad (4.25)$$

below T_c . At T_c there is no discontinuity in entropy, or latent heat, confirming that the GL model corresponds to a second-order thermodynamic phase transition. But there is a sudden change in specific heat at T_c . Differentiating the entropy to find the heat capacity $C_V = Tds/dT$ per unit volume we obtain

$$C_{Vs} - C_{Vn} = \begin{cases} T\dot{a}^2/b & T < T_c, \\ 0 & T > T_c. \end{cases} \quad (4.26)$$

and so the heat capacity has a discontinuity

$$\Delta C_V = T_c \frac{\dot{a}^2}{b} \quad (4.27)$$

at T_c . The metallic normal state heat capacity is linear in T , $C_{Vn} = \gamma T$, with γ the Sommerfeld constant, and so the full heat capacity curve looks like Fig. 4.4 near to T_c .³

Interestingly the specific heat for superconductors shown in Fig. 4.4 is qualitatively quite very different from both the case of BEC, shown in Fig. 1.7, and the λ point of superfluid ⁴He, Fig. 2.3.⁴

4.4 Ginzburg–Landau theory of inhomogenous systems

The complete GL theory of superconductivity also allows for the possibility that the order parameter depends on position, $\psi(\mathbf{r})$. This, of course, now really begins to resemble the macroscopic condensate wave function introduced in Chapter 2 for the case of superfluid helium.

GL postulated that the free energy is as given above, together with a new term depending on the gradient of $\psi(\mathbf{r})$. With this term the free energy density becomes,

$$f_s(T) = f_n(T) + \frac{\hbar^2}{2m^*} |\nabla \psi(\mathbf{r})|^2 + a(T) |\psi(\mathbf{r})|^2 + \frac{b(T)}{2} |\psi(\mathbf{r})|^4 \quad (4.28)$$

at point $\mathbf{r}$ in the absence of any magnetic fields. Setting $\psi(\mathbf{r})$ to a constant value, $\psi(\mathbf{r}) = \psi$, we see that the parameters $a(T)$ and $b(T)$ are the same as for the bulk theory described in the previous section. The new parameter m^* determines the energy cost associated with gradients in $\psi(\mathbf{r})$. It has dimensions of mass, and it plays the role of an effective mass for the quantum system with macroscopic wave function $\psi(\mathbf{r})$.

In order to find the order parameter $\psi(\mathbf{r})$ we must minimize the total free energy of the system,

$$F_s(T) = F_n(T) + \int \left(\frac{\hbar^2}{2m^*} |\nabla \psi|^2 + a(T) |\psi(\mathbf{r})|^2 + \frac{b(T)}{2} |\psi(\mathbf{r})|^4 \right) d^3r. \quad (4.29)$$

To find the minimum we must consider an infinitesimal variation in the function $\psi(\mathbf{r})$

$$\psi(\mathbf{r}) \rightarrow \psi(\mathbf{r}) + \delta\psi(\mathbf{r}) \quad (4.30)$$

relative to some function $\psi(\mathbf{r})$. Evaluating the change in the total free energy due to $\delta\psi$ and dropping all terms of higher than linear order in the variation $\delta\psi$ we find after some lengthy algebra

$$\begin{aligned}\delta F_s = & \int \left[\frac{\hbar^2}{2m^*} (\nabla \delta\psi^*) \cdot (\nabla \psi) + \delta\psi^* (a\psi + b\psi|\psi^2|) \right] d^3r \\ & + \int \left[\frac{\hbar^2}{2m^*} (\nabla \psi^*) \cdot (\nabla \delta\psi) + (a\psi^* + b\psi^*|\psi^2|) \delta\psi \right] d^3r. \quad (4.31)\end{aligned}$$

The two terms involving gradients can be integrated by parts, to obtain

$$\begin{aligned}\delta F_s = & \int \delta\psi^* \left(-\frac{\hbar^2}{2m^*} \nabla^2 \psi + a\psi + b\psi|\psi^2| \right) d^3r \\ & + \int \left(-\frac{\hbar^2}{2m^*} \nabla^2 \psi^* + a\psi^* + b\psi^*|\psi^2| \right)^* \delta\psi d^3r. \quad (4.32)\end{aligned}$$

The condition for $\psi(\mathbf{r})$ to produce a minimum in free energy is that $\delta F = 0$ for any arbitrary variation $\delta\psi(\mathbf{r})$. From Eq. 4.32 this can only be when $\psi(\mathbf{r})$ obeys

$$-\frac{\hbar^2}{2m^*} \nabla^2 \psi + a\psi + b\psi|\psi^2| = 0. \quad (4.33)$$

We can obtain this same result more formally by noting that the total free energy of the solid is a **functional** of $\psi(\mathbf{r})$, denoted $F_s[\psi]$, meaning that the scalar number F_s depends on the whole function $\psi(\mathbf{r})$ at all points in the system, $\mathbf{r}$. It will be minimized by a function $\psi(\mathbf{r})$ which satisfies

$$\frac{\partial F_s[\psi]}{\partial \psi(\mathbf{r})} = 0 \quad \frac{\partial F_s[\psi]}{\partial \psi^*(\mathbf{r})} = 0. \quad (4.34)$$

where the derivatives are mathematically **functional derivatives**. Functional derivative can be defined by analogy with the idea of a partial derivative. For a function of many variables, $f(x_1, x_2, x_3, \dots)$ we can express changes in the function value due to infinitesimal variations of the parameters using the standard expression

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 + \dots \quad (4.35)$$

Considering the free energy as a function of infinitely many variables, $\psi(\mathbf{r})$ and $\psi^*(\mathbf{r})$ at all possible points $\mathbf{r}$ we can write the analog of Eq. 4.35 as,

$$dF_s = \int \left(\frac{\partial F_s[\psi]}{\partial \psi(\mathbf{r})} d\psi(\mathbf{r}) + \frac{\partial F_s[\psi]}{\partial \psi^*(\mathbf{r})} d\psi^*(\mathbf{r}) \right) d^3r. \quad (4.36)$$

In comparison with Eq. 4.32 we see that

$$\frac{\partial F_s[\psi]}{\partial \psi^*(\mathbf{r})} = -\frac{\hbar^2}{2m^*} \nabla^2 \psi + a(T)\psi + b(T)\psi|\psi^2| \quad (4.37)$$

and

$$\frac{\partial F_s[\psi]}{\partial \psi(\mathbf{r})} = \left(-\frac{\hbar^2}{2m^*} \nabla^2 \psi^* + a(T)\psi^* + b(T)\psi^*|\psi^2| \right)^*, \quad (4.38)$$

which is just the complex conjugate of Eq. 4.37. Perhaps it seems surprising that we can effectively treat $\psi(\mathbf{r})$ and $\psi^*(\mathbf{r})$ as independent variables in the differentiation, but this is correct because there are two independent real functions, $\text{Re}[\psi(\mathbf{r})]$ and $\text{Im}[\psi(\mathbf{r})]$, which can be varied separately.

Thus we have found that minimizing the total free energy leads to the following Schrödinger like equation for $\psi(\mathbf{r})$,

$$-\frac{\hbar^2}{2m^*}\nabla^2\psi(\mathbf{r}) + (a + b|\psi(\mathbf{r})|^2)\psi(\mathbf{r}) = 0. \quad (4.39)$$

However, unlike the usual Schrödinger equation, this is a nonlinear equation because of the second term in the bracket. Because of this nonlinearity the quantum mechanical principle of superposition does not apply, and the normalization of ψ is different from the usual one in quantum mechanics.

4.5. Ginzburg–Landau theory in a magnetic field

The full power of the GL approach to superconductors only becomes apparent when we include one final term, the effect of a magnetic field. It is this which truly shows that this is a fully fledged theory of superconductivity, complete with Meissner–Ochsenfeld effect, London equation, and so on. Effectively, the GL theory as developed given in the previous sections did not include any effects of the charge of the superconducting condensate. Therefore it would be appropriate for systems of neutral particles, such as a superfluid, or for situations where there are no supercurrents. But in the presence of supercurrents of charged particles we must extend the theory to include the interaction of the current and magnetic field.

What is needed is to include the effects of magnetic fields in the free energy. GL postulated that the magnetic field enters as if $\psi(\mathbf{r})$ were the wave function for charged particles, that is, with the usual replacement in quantum mechanics

$$\frac{\hbar}{i}\nabla \rightarrow \frac{\hbar}{i}\nabla - q\mathbf{A} \quad (4.47)$$

where q is the charge and $\mathbf{A}$ is the magnetic vector potential. For all known superconductors it turns out that the appropriate charge q is $-2e$. Why this is the case only became clear after the BCS theory was developed, and the link

⁵In fact, in their original paper GL assumed that the effective charge would be e , not $2e$. Reputedly, Ginzburg, then a young researcher, told his famous advisor Landau that $2e$ fitted the available experimental data better than e , but Landau overruled him and insisted that he was sure that it must be e ! Ginzburg shared the 2003 Nobel Prize in Physics for his work in developing the GL model.

⁶Note that in the definition of the magnetic work given in Eq. 4.4, we excluded the part of this field energy, $\mu_0 H^2/2$ per unit volume, that would be present even with no sample present inside the coil in Fig. 3.6. From now on, it will be more convenient to include this energy explicitly, making F_s the total free energy of both sample and vacuum fields.

between the BCS theory and the GL model had been explained by Gor'kov. He showed that the correct physical interpretation of the GL order parameter $\psi(\mathbf{r})$ is that it can be understood as the wave function for the center of mass motion of Cooper pairs of electrons. Since each Cooper pair has a net charge of $-2e$, then this is the correct effective charge q .⁵ Actually the sign can equally well be taken as $q = +2e$, since we can think of Cooper pairs of holes as readily as pairs of electrons. In fact no observable effects in the GL theory differ when we take a different convention for the sign.

With this replacement the GL free energy density of the superconductor becomes,

$$f_s(T) = f_n(T) + \frac{\hbar^2}{2m^*} \left| \left(\frac{\hbar}{i} \nabla + 2e\mathbf{A} \right) \psi \right|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4. \quad (4.48)$$

To obtain the total free energy we must integrate this over the system, but we must also include an additional term, corresponding to the electromagnetic field energy of the field $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}$ at each point $\mathbf{r}$. Therefore the total free energy of both the superconductor and the magnetic field is,⁶

$$F_s(T) = F_n(T) + \int \left(\frac{\hbar^2}{2m^*} \left| \left(\frac{\hbar}{i} \nabla + 2e\mathbf{A} \right) \psi \right|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 \right) d^3r + \frac{1}{2\mu_0} \int \mathbf{B}(\mathbf{r})^2 d^3r. \quad (4.49)$$

The first integral is carried out over points $\mathbf{r}$ inside the sample, while the second is performed over all space.

The condition for the minimum free energy state is again found by performing a functional differentiation to minimize with respect to $\psi(\mathbf{r})$ and $\psi^*(\mathbf{r})$. The resulting equation for $\psi(\mathbf{r})$ is again a nonlinear Schrödinger equation, but now with a term containing the magnetic vector potential $\mathbf{A}$,

$$-\frac{\hbar^2}{2m^*} \left(\nabla + \frac{2ei}{\hbar} \mathbf{A} \right)^2 \psi(\mathbf{r}) + (a + b|\psi|^2) \psi(\mathbf{r}) = 0. \quad (4.50)$$

The supercurrents due to the magnetic field can be found from the functional derivative of the GL superconductor free energy with respect to the vector potential,

$$\mathbf{j}_s = -\frac{\partial F_s}{\partial \mathbf{A}(\mathbf{r})} \quad (4.51)$$

which leads to the supercurrent

$$\mathbf{j}_s = -\frac{2e\hbar i}{2m^*} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{(2e)^2}{m^*} |\psi|^2 \mathbf{A}. \quad (4.52)$$

Note the close similarity to the superfluid current flow that we found earlier in the case of ^4He , Eq. 2.21. The differences from Eq. 2.21 are first the charge of the condensate particles, $-2e$, and the presence of the last term which provides the effect of the vector potential $\mathbf{A}$. Finally, the vector potential must be obtained from the magnetic field arising from both the supercurrents and any other currents, such as the external currents, $\mathbf{j}_{\text{ext}}$, in the solenoid coils of Fig. 3.6,

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{j}_{\text{ext}} + \mathbf{j}_s), \quad (4.53)$$

as given by Maxwell's equations.

NB. In the formula (4.52) the charge is negative

We have now two GL equations which along with boundary conditions provide a full set of differential equations to solve. These equations immediately lead us to important physical properties of superconductors.

A link for the textbook by Schmidt “The Physics of Superconductors. Introduction to Fundamentals and Applications” <https://link.springer.com/book/10.1007/978-3-662-03501-6>

Hour 6.

4.6. Coherence length.

Let us consider a homogeneous SC $\psi(\vec{r}) = \psi_0 = \sqrt{\frac{n_s}{2}} = \sqrt{\frac{|\alpha|}{\beta}}$ everywhere except a weaker region in which the OP is not as strong as others regions of the same superconductor. No magnetic field is applied. We try to see at which spatial scale the OP evolves from this (weaker region) to a normal (ψ_0) OP. Since the depletion of the order parameter is weak we can write $\psi(\vec{r}) = \psi_0(1 - \varepsilon(\vec{r}))$ where $\varepsilon(\vec{r}) \ll 1$. For simplicity we will consider the planar geometry, $\psi(x) = \psi_0(1 - \varepsilon(x))$. The weak location is situated at $x = 0$ Let us put this to the first GL equation:

$$-\frac{\hbar^2}{2m^*} \frac{d^2}{dx^2} \psi(x) + \alpha \psi(x) + \beta |\psi(x)|^2 \psi(x) = 0$$

$$\psi_0 \frac{\hbar^2}{2m^*} \frac{d^2}{dx^2} \varepsilon(x) + \alpha \psi_0(1 - \varepsilon(x)) + \beta \psi_0^3(1 - \varepsilon(x))^3 = 0$$

Since $\varepsilon(x) \ll 1$

$$\psi_0 \frac{\hbar^2}{2m^*} \frac{d^2}{dx^2} \varepsilon(x) + \alpha \psi_0(1 - \varepsilon(x)) + \beta \psi_0^3(1 - 3\varepsilon(x)) = 0$$

If we define $\xi^2 = \frac{\hbar^2}{2m^*|\alpha|}$ and $\psi_0 = \sqrt{\frac{|\alpha|}{\beta}}$, we get (since $\alpha + \beta \psi_0^2 + (-\alpha \varepsilon(x) + 3\beta \psi_0^2 \varepsilon(x)) = 0 + 2\alpha \varepsilon(x) = 2\alpha \varepsilon(x)$)

$$\xi^2 \frac{d^2}{dx^2} \varepsilon(x) - 2\varepsilon(x) = 0$$

with the condition $\varepsilon(x \rightarrow \infty) = 0$. The solution of this equation is

$$\varepsilon(x) = \varepsilon(0) e^{-\frac{\sqrt{2}x}{\xi}}$$

i.e. the recovery of the OP locally weakened at $x = 0$ is not immediate but takes place on the scale of ξ .

NB. ξ is called “coherence length”. It depends on temperature (diverges at $T \rightarrow T_c$) as

$$\xi(T) = \left(\frac{\hbar^2}{2m^*|\alpha(T)|} \right)^{1/2} = \left(\frac{\hbar^2}{2m^*\tilde{a}(T_c - T)} \right)^{1/2}$$

This scale (1-100nm) is very important as it gives, in addition to London/GL penetration depth $\lambda_L = \left(\frac{m_e}{\mu_0 e^2 n_s}\right)^{1/2} \approx \lambda_{GL} = \left(\frac{m^* \beta}{2\mu_0 e^2 |\alpha|}\right)^{1/2} = \left(\frac{m^* b}{2\mu_0 e^2 \tilde{a}(T_C - T)}\right)^{1/2}$ (at $T \rightarrow T_C$) the second characteristic scale of SC.

NB. We see that while both $\xi(T)$ and $\lambda(T)$ diverge at $T \rightarrow T_C$, their ratio does not:

$$\kappa = \lambda/\xi$$

The parameter κ is very important as it decides the type of SC.

NB. A similar example provided by J. F. Annett and presented below is not physically correct, as there is no physical meaning of putting the OP to zero at the S-N interface. Rather, at S-N interface there is so-called “proximity effect” that we will discuss in the next section.

Consider a simple model for the interface between a normal metal and a superconductor. Suppose that the interface lies in the yz plane separating the normal metal in the $x < 0$ region from the superconductor in the $x > 0$ region. On the normal metal side of the interface the superconducting order parameter, $\psi(\mathbf{r})$, must be zero. Assuming that $\psi(\mathbf{r})$ must be continuous, we must therefore solve the nonlinear Schrödinger equation,

$$-\frac{\hbar^2}{2m^*} \frac{d^2\psi(x)}{dx^2} + a(T)\psi(x) + b(T)\psi^3(x) = 0 \quad (4.40)$$

in the region $x > 0$ with the boundary condition at $\psi(0) = 0$. It turns out that one can solve this equation directly (Exercise 4.2) to find

$$\psi(x) = \psi_0 \tanh\left(\frac{x}{\sqrt{2}\xi(T)}\right), \quad (4.41)$$

as shown in Fig. 4.5. Here ψ_0 is the value of the order parameter in the bulk far from the surface and the parameter $\xi(T)$ is defined by

$$\xi(T) = \left(\frac{\hbar^2}{2m^*|a(T)|}\right)^{1/2}. \quad (4.42)$$

This quantity, which has dimensions of length, is called the **Ginzburg–Landau coherence length**. It is an important physical parameter characterizing the superconductor. In Fig. 4.5 one can see that $\xi(T)$ is a measure of the distance from the surface over which the order parameter has recovered back to nearly its bulk value.

The GL coherence length arises in almost all problems of inhomogeneous superconductors, including surfaces, interfaces, defects, and vortices. Using $a(T) = \tilde{a}(T - T_C)$ the coherence length $\xi(T)$ can be rewritten,

$$\xi(T) = \xi(0)|t|^{-1/2}, \quad (4.43)$$

where

$$t = \frac{T - T_C}{T_C} \quad (4.44)$$

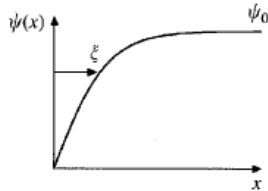


Fig. 4.5 Order parameter of a superconductor near a surface. It recovers to its bulk value ψ_0 over a length scale of the coherence length, ξ .

4.7. Proximity effect

The strength of GL theory is also to predict the so-called proximity effect which is the building block of Josephson junctions. In fact, when a superconductor is brought in a (very good) contact to a non

superconducting material (metal, semiconductor or even insulator) the superconducting electrons can leak outside and induce quantum coherent state in the vicinity of the interface. The scale of this effect strongly depends on materials used. In general, the proper description requires microscopic theories (BCS, Usadel etc.) and cannot be done within GL. Here we will give a specific example: a junction between two superconductors with different critical temperatures, (T_{cn} and T_{cs}), brought in good contact: $T_{cs} > T_{cn}$, $T_{cs} - T_{cn} \ll T_{cn}$. The temperature of the specimen T is chosen to satisfy the inequality $T_{cn} < T < T_{cs}$, hence the material with the critical temperature T_{cn} is in the normal state. The interface between the two materials is flat and coincides with the plane $x = 0$. The superconductor occupies the semispace $x > 0$, and the “normal” metal $x < 0$. The behavior of the order parameter in the S region ($x > 0$) can be determined by solving the first GL equation (reduced form $\psi(\vec{r}) \rightarrow \psi(\vec{r})/\psi_0$ with $\psi_0 = \sqrt{\frac{|\alpha|}{\beta}}$)

$$-\frac{\hbar^2}{2m^*} \left(\vec{\nabla} + \frac{i2e}{\hbar} \vec{A} \right)^2 \psi(\vec{r}) - |\alpha|\psi(\vec{r}) + \beta|\psi(\vec{r})|^2\psi(\vec{r}) = 0$$

$$-\frac{\hbar^2}{2m^*} \sqrt{\frac{|\alpha|}{\beta}} \left(\vec{\nabla} + \frac{i2e}{\hbar} \vec{A} \right)^2 \psi(\vec{r}) - \sqrt{\frac{|\alpha|}{\beta}} |\alpha|\psi(\vec{r}) + \sqrt{\frac{|\alpha|}{\beta}} \frac{|\alpha|}{\beta} \beta|\psi(\vec{r})|^2\psi(\vec{r}) = 0$$

$$-\frac{\hbar^2}{2m^*} \left(\vec{\nabla} + \frac{i2e}{\hbar} \vec{A} \right)^2 \psi(\vec{r}) - |\alpha|\psi(\vec{r}) + |\alpha||\psi(\vec{r})|^2\psi(\vec{r}) = 0$$

By dividing by $|\alpha|$:

$$-\frac{\hbar^2}{2m^*} \frac{1}{|\alpha|} \left(\vec{\nabla} + \frac{i2e}{\hbar} \vec{A} \right)^2 \psi(\vec{r}) - \psi(\vec{r}) + |\psi(\vec{r})|^2\psi(\vec{r}) = 0$$

$$-\xi^2 \left(\vec{\nabla} + \frac{i2e}{\hbar} \vec{A} \right)^2 \psi(\vec{r}) - \psi(\vec{r}) + |\psi(\vec{r})|^2\psi(\vec{r}) = 0$$

Now, in one dimension, and considering ψ real,

$$-\xi^2 d^2\psi(x)/dx^2 - \psi(x) + \psi^3(x) = 0$$

Which is possible to integrate it directly:

$$-\xi^2 \left(\frac{d\psi(x)}{dx} \right)^2 - \psi^2(x) + \frac{1}{2} \psi^4(x) = Cte$$

$$\left(\text{since } \frac{d\left(\frac{d\psi(x)}{dx}\right)^2}{dx} = 2 \frac{d^2}{dx^2} \psi(x) \frac{d\psi}{dx}, \quad \frac{d(\psi^2)}{dx} = 2\psi(x) \frac{d\psi}{dx}, \quad \text{and } \frac{d\left(\frac{1}{2}\psi^4\right)}{dx} = 2\psi^3(x) \frac{d\psi}{dx} \right)$$

Since at $(x \rightarrow \infty)$ we have $\frac{d\psi(x)}{dx} \rightarrow 0$ and $\psi(x) \rightarrow 1$, the integration constant is $Cte = -1/2$.

Substituting this into the equation and carrying out the integration, we obtain

$$\psi(x) = \tanh((x - x_0)/\sqrt{2}\xi)$$

Here x_0 is the integration constant to be determined from the boundary condition at $x = 0$: no current should spontaneously flow through the boundary. In our case this condition leads to $x_0 < 0$ (i.e. the OP is still not zero at $x = 0$). The precise value of x_0 , in general, should be calculated using the microscopic theory.

Its geometrical significance becomes evident from Figure 3.2.

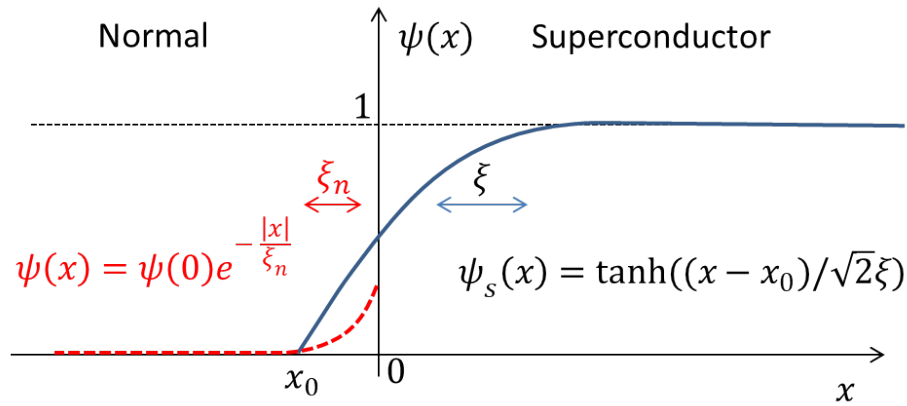
Consider now the behavior of the order parameter $\psi(x)$ in the normal region $x < 0$. Since the material is just above its SC transition, $\alpha_n > 0$ and the first GL equation reads

$$-\xi_n^2 \frac{d^2}{dx^2} \psi(x) + \psi(x) + \psi^3(x) = 0$$

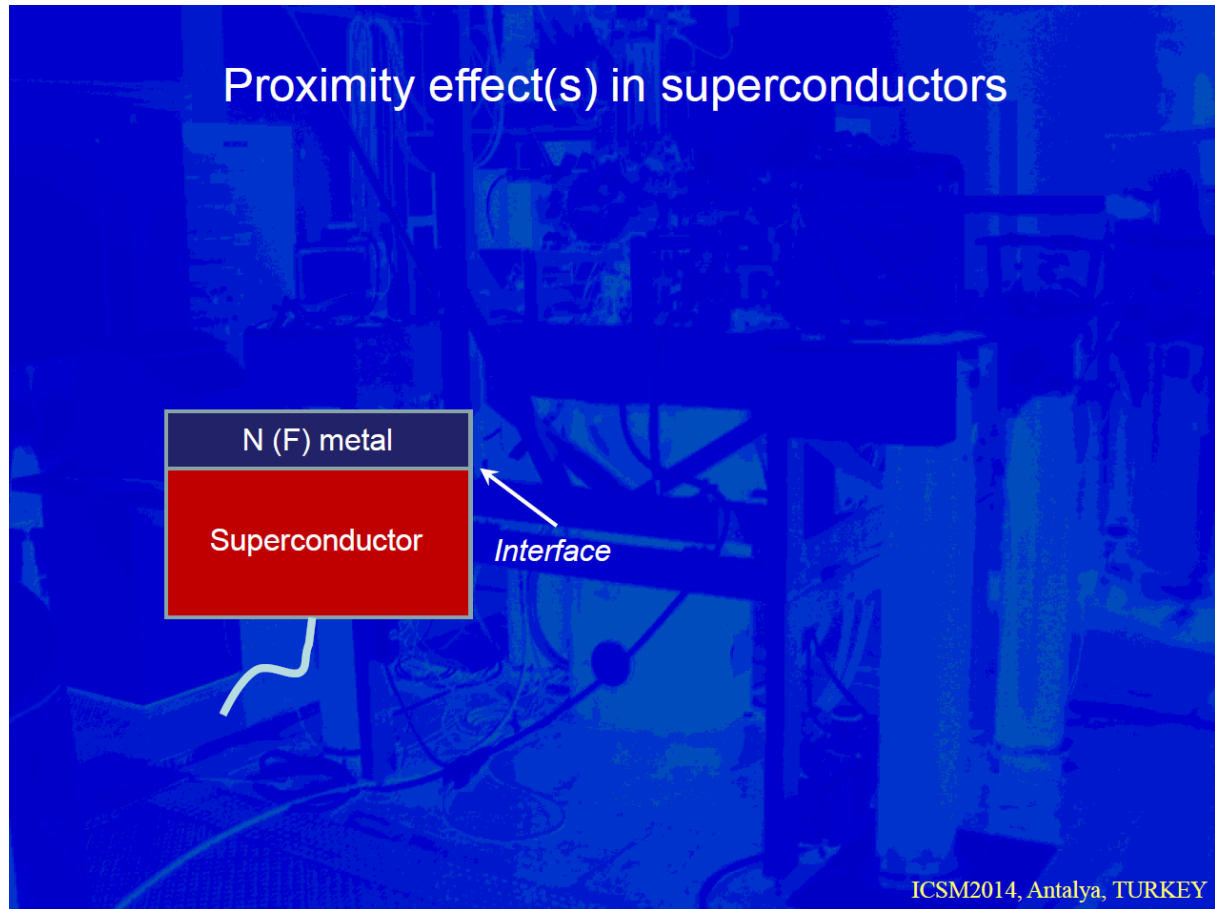
With $\xi_n^2 = \frac{\hbar^2}{2m^*|\alpha_n|}$. Since the induced OP is very small the cubic term can be neglected. Then, taking the limit conditions $\psi(x \rightarrow -\infty) = 0$ we get the solution

$$\psi(x) = \psi(0)e^{-\frac{|x|}{\xi_n}}$$

The “normal” coherence length depends on electronic properties of the material. It varies from 1nm (in insulators) up to microns in good metals with high mean free path $l \gg \xi_n$.



4.8. Power Point presentation on Proximity effect as observed by STM.



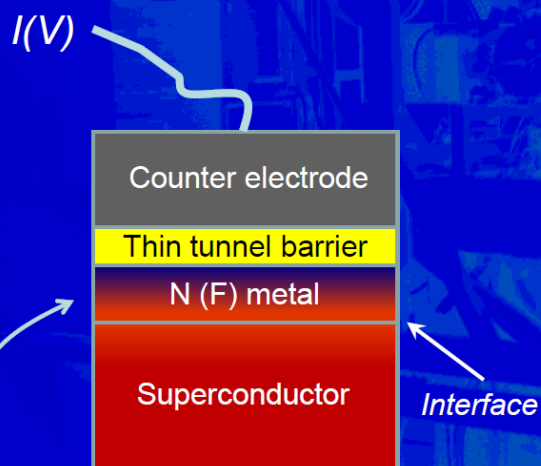
Proximity effect(s) in superconductors



N(F) thickness (spatial) dependence is crucial

ICSM2014, Antalya, TURKEY

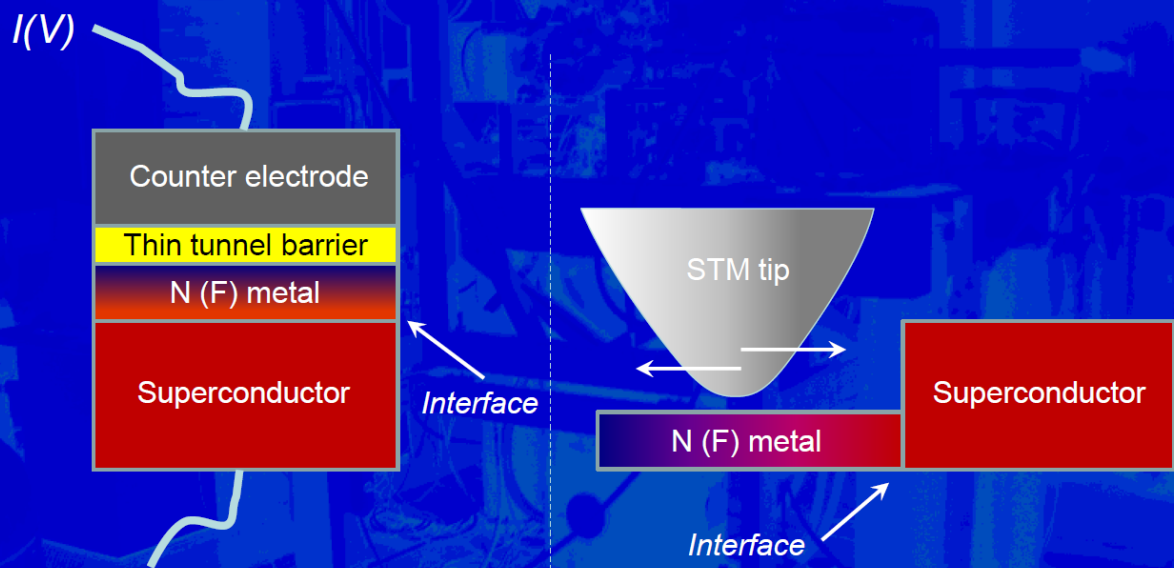
Proximity effect(s) in superconductors



N(F) thickness (spatial) dependence is crucial

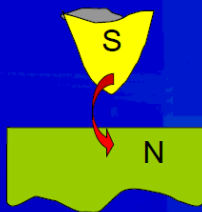
ICSM2014, Antalya, TURKEY

Proximity effect(s) in *lateral* junctions



Advantage: Direct information about spatial evolution of the DOS

ICSM2014, Antalya, TURKEY



Tunneling spectroscopy of superconductors: *Nb*

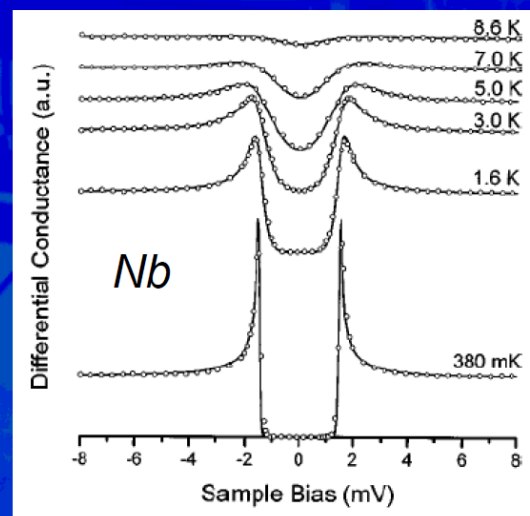
$$dI / dV(\mathbf{r}) \propto \int_{-\infty}^{\infty} N_s(E, \mathbf{r}) \left[\frac{-\partial f(E + eV)}{\partial(eV)} \right] dE$$

$$E_k = \sqrt{(\varepsilon_k)^2 + (\Delta_k)^2}$$

$$N_S(E) = N_N(E) \frac{E}{\sqrt{E^2 - \Delta^2}}$$

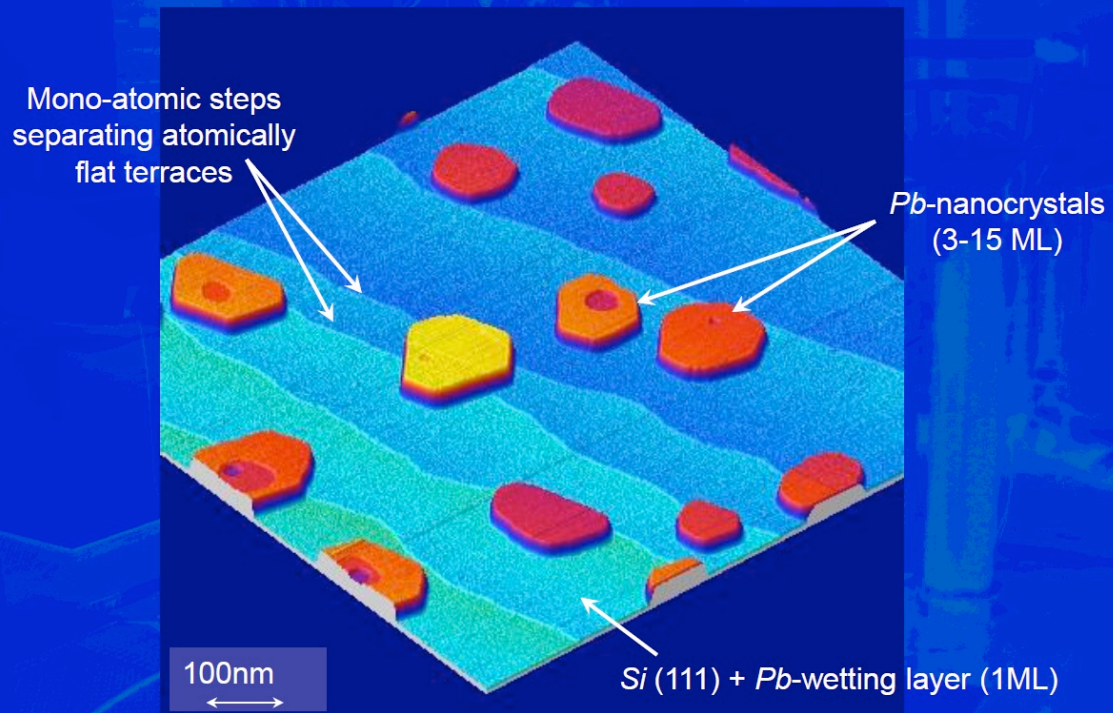
The gap closes at $T_c=9.2\text{K}$

Pan et al. APL, 73 (1998)



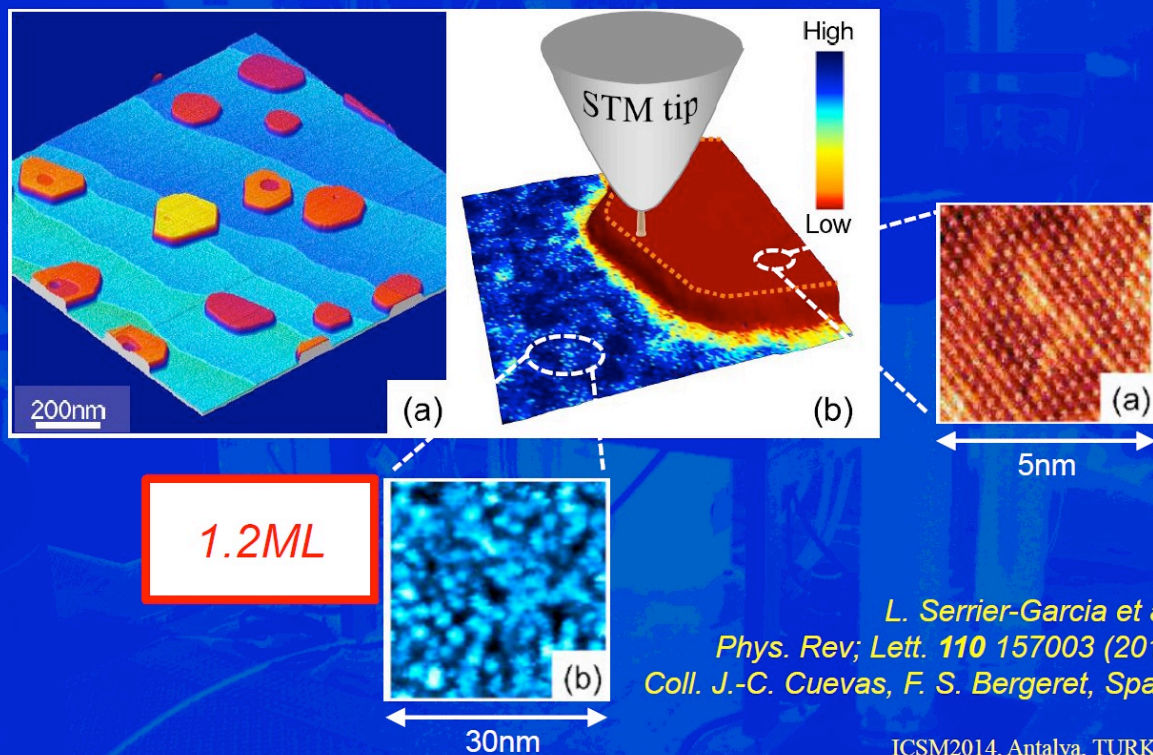
ICSM2014, Antalya, TURKEY

Pb in-situ grown on 7x7 Si(111) in UHV



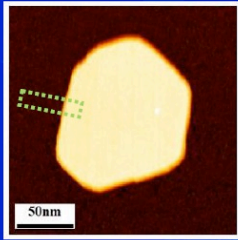
ICSM2014, Antalya, TURKEY

Proximity effect between Pb-island and wetting layer



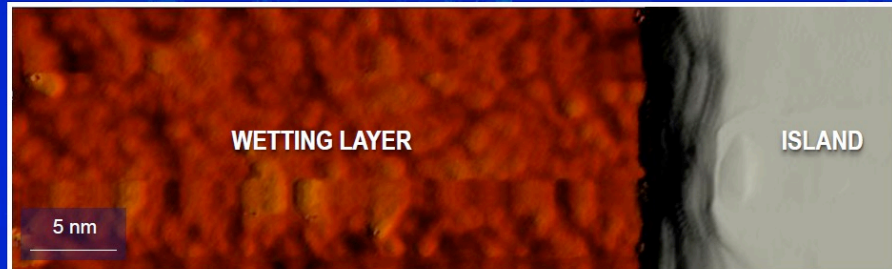
L. Serrier-Garcia et al.
Phys. Rev. Lett. **110** 157003 (2013)
Coll. J.-C. Cuevas, F. S. Bergeret, Spain

ICSM2014, Antalya, TURKEY

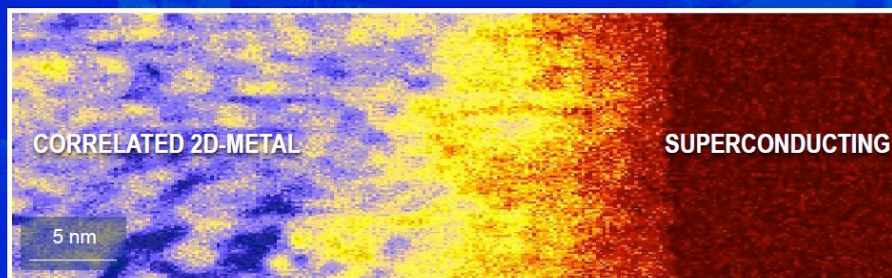


Proximity effect between Pb-island and wetting layer

Topography



Spectroscopy zero Bias



4 nm



0 nm

36 nS



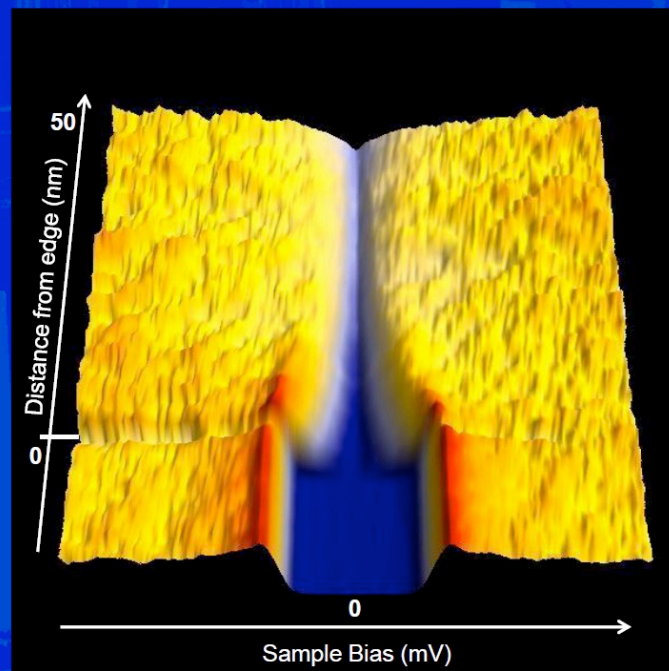
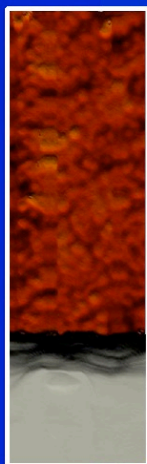
0 16/28

Serrier-Garcia, PRL 110, 157003 (2013)

ICSM2014, Antalya, TURKEY

Proximity effect between Pb-island and wetting layer

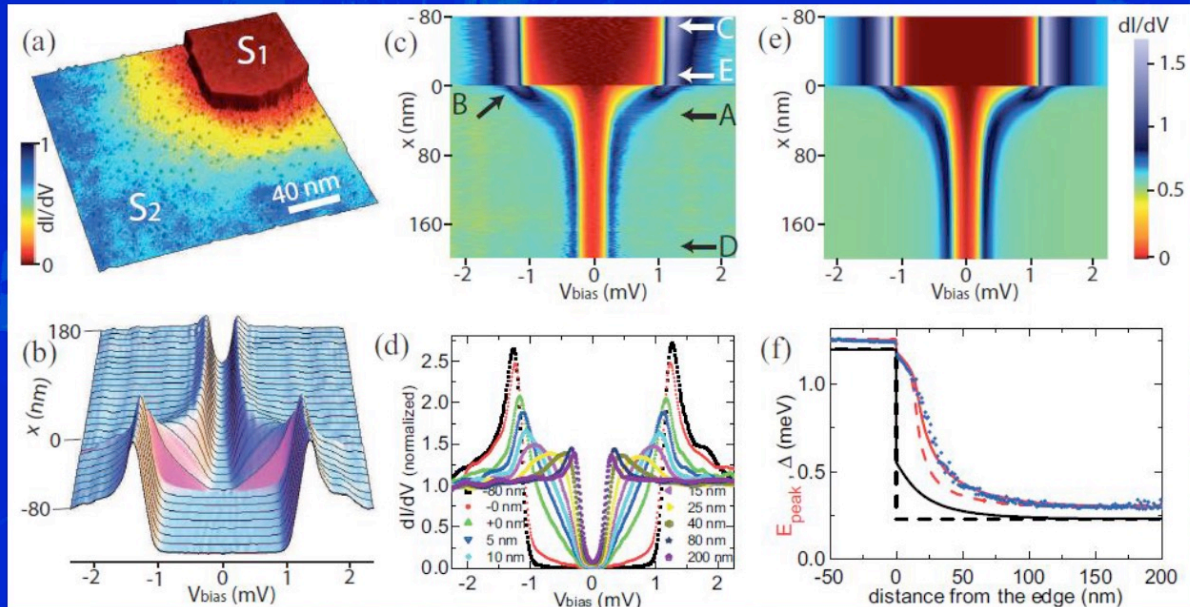
Topography



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Proximity effect in lateral S_1 - S_2 junctions

Low temperature regime: $T < T_{C1}, T_{C2}$

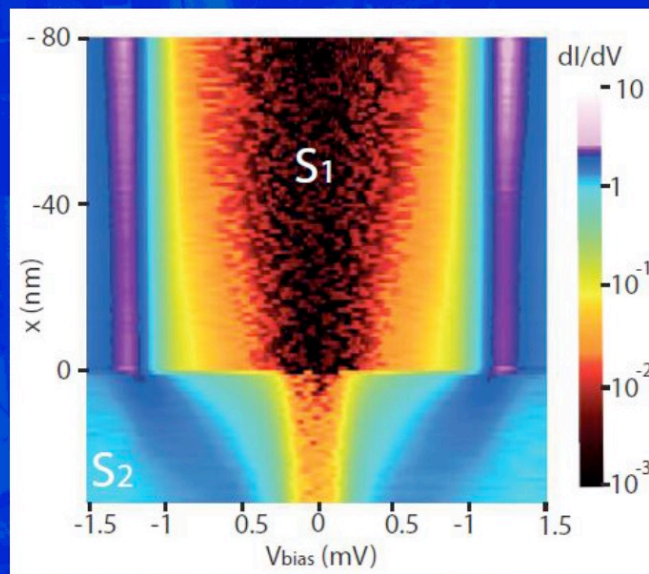


(Phys. Rev. X, 2014)

ICSM2014, Antalya, TURKEY

Proximity effect in lateral S_1 - S_2 junctions

Revealing the inverse proximity effect in S_1

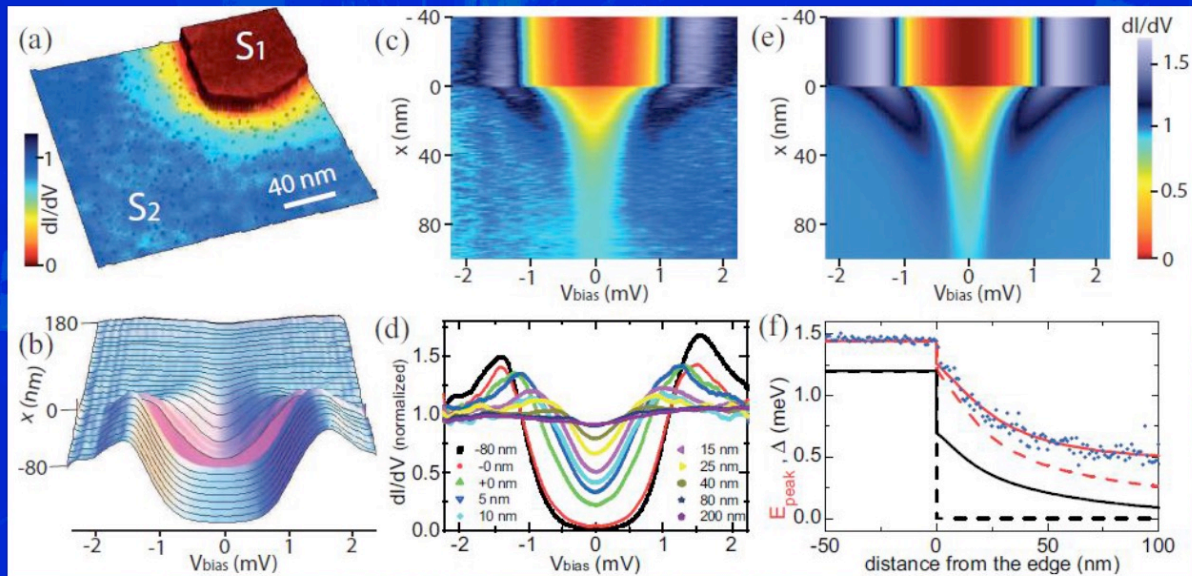


(Phys. Rev. X, 2014)

ICSM2014, Antalya, TURKEY

Proximity effect in lateral S_1 - S_2 junctions

« High » temperature regime: $T_{C2} < T < T_{C1}$

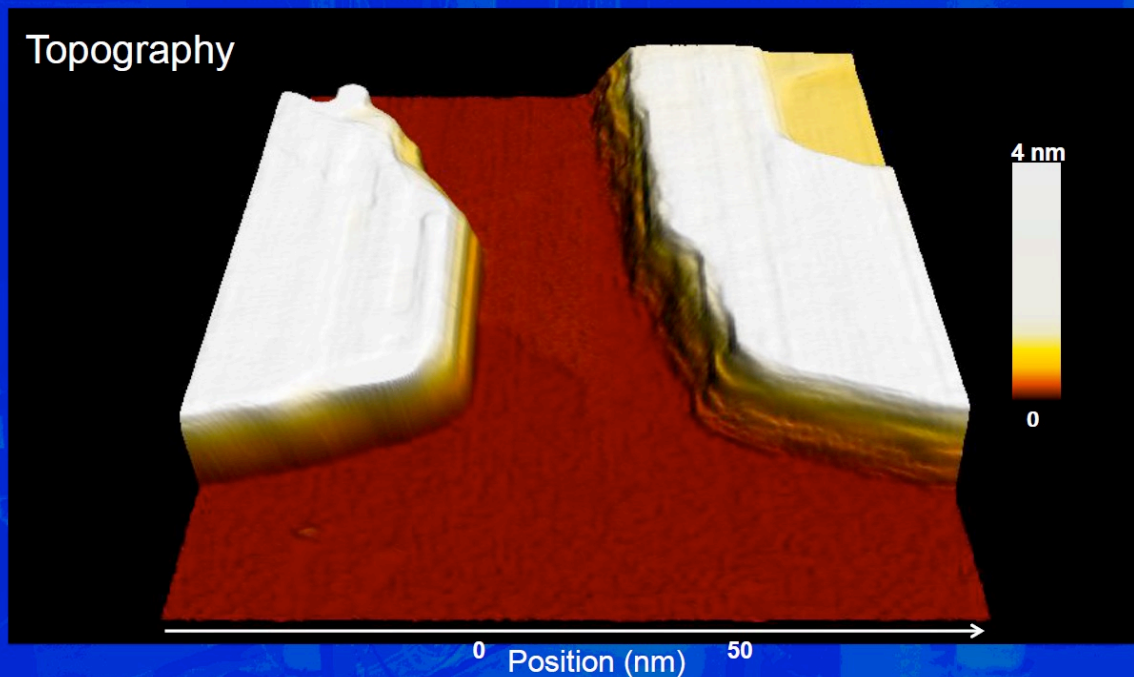


deGennes' prediction (1968) – experimentally demonstrated 2013
(*Phys. Rev. X*, 2014)

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Elaborating lateral SNS proximity junctions

Topography

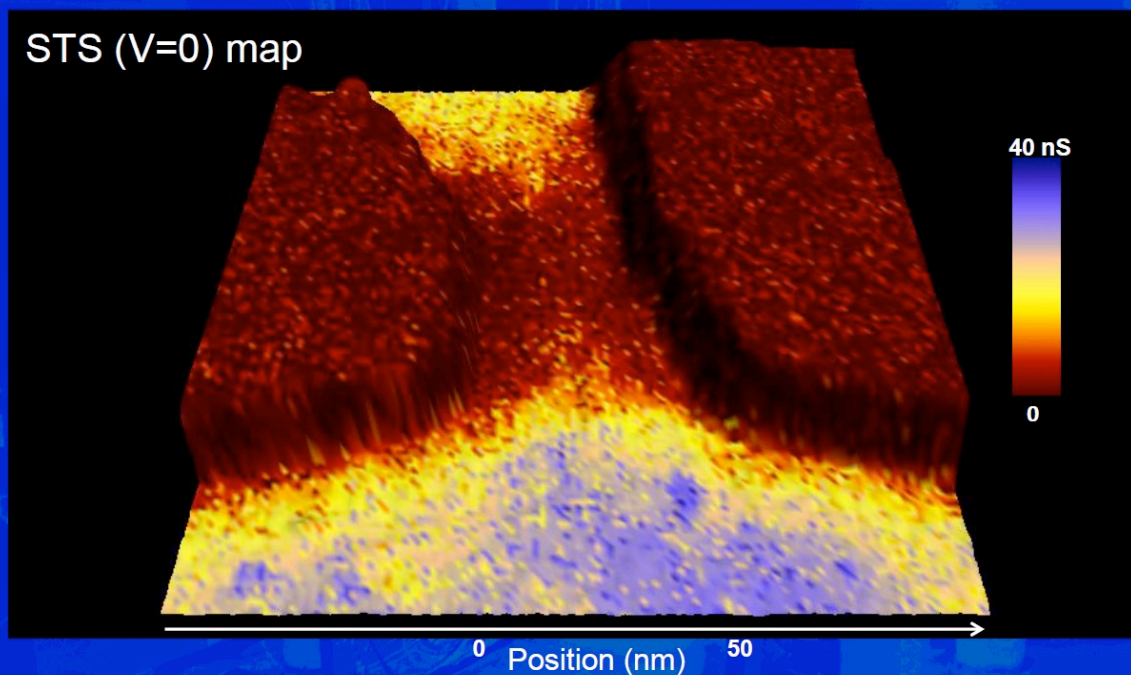


Proximity SNS junctions can be obtained by incrising the density of islands

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Elaborating lateral SNS proximity junctions

STS ($V=0$) map

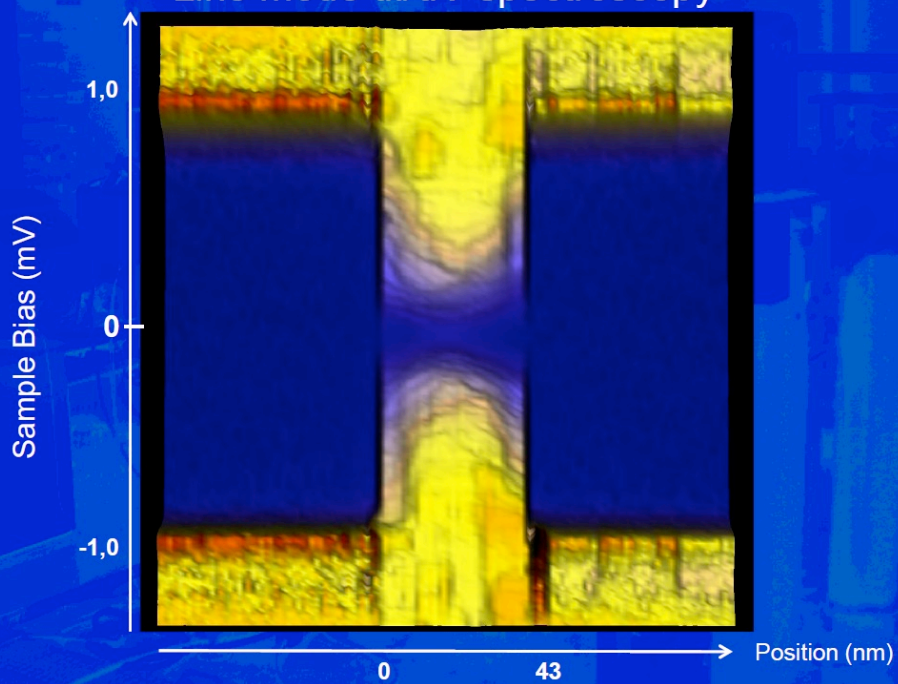


Proximity link between two Pb-islands

ICSM2014, Antalya, TURKEY

Elaborating lateral SNS proximity junctions

Line mode dI/dV spectroscopy



The mini-gap exists in the proximity region

ICSM2014, Antalya, TURKEY

Hour 7.

4.9. Flux quantization within the GL formalism.

With respect to London equation,

$$\vec{J}_s = -\frac{c}{4\pi\lambda_L^2} \vec{A}, \lambda_L = \left[\frac{mc^2}{4\pi n_s e^2} \right]^{1/2} \text{ (CGS)}$$

or

$$\vec{J}_s = -\frac{1}{\mu_0 \lambda_L^2} \vec{A}, \lambda_L = \left[\frac{m}{\mu_0 n_s e^2} \right]^{1/2} \text{ (SI)}$$

$$j_s = -\frac{(2e)^2}{m^*} n_s \vec{A}$$

Second GL equation has a similar term:

$$-\frac{(2e)^2}{m^*} |\psi|^2 \vec{A}$$

which immediately tells us about the meaning of $|\psi|^2$. But, very importantly, it also has an additional QM term related to the gradients of the wave function:

$$-i \frac{(2e)\hbar}{2m^*} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$$

It tells us that supercurrents could flow owing gradients of the OP (but only if the OP is complex, i.e. contains a phase!!!).

Case 1: macroscopic SC sample with a macroscopic hole (similar to what we already considered, see hour 3).

Let us consider a macroscopic superconductor with a hole drilled inside. Let us chose a closed loop inside the material far from the hole ($\gg \lambda_L$). From the London equation we get on the loop

$$j_s^{loop} = 0, H^{loop} = 0$$

Consider second GL equation

$$j_s = -i \frac{(2e)\hbar}{2m^*} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) - \frac{(2e)^2}{m^*} |\psi|^2 \vec{A}$$

Since the loop is far from the hole amplitude of the order parameter does not depend on position, only the phase could depend: $\psi(\theta(r)) = |\psi| e^{i\theta(r)}$

$$j_s = \frac{(2e)^2}{m^*} |\psi|^2 \left[\frac{\hbar}{2e} \vec{\nabla} \theta - \vec{A} \right]$$

Since on the loop $j_s = 0$, this results in: $\frac{\hbar}{2e} \vec{\nabla} \theta = \vec{A}$.

Let us now take a path integral from both parts over the loop. We get

$$\frac{\hbar}{2e} \oint d\vec{s} \vec{\nabla} \theta = \oint d\vec{s} \vec{A}$$

By Stoke's theorem

NB: Anecdote about the Stoke's theorem. Lord Kelvin communicated it to George Stokes in a letter dated July 2, 1850 (without demonstration). Stokes set the theorem as a question on the 1854 Smith's Prize exam, which led to the result bearing his name, even though it was actually first published by Hermann Hankel in 1861.

$$\oint d\vec{s} \vec{A} = \int d\vec{S} \text{rot}(\vec{A}) = \int d\vec{S} \vec{B} = \Phi$$

Φ being the magnetic flux across the loop. Thus we have

$$\frac{\hbar}{2e} \oint d\vec{s} \vec{\nabla} \theta = \Phi$$

One immediately sees that θ is not uni-defined, but $\psi(\vec{r})$ should be. Finally, if the phase accumulation along the loop is $2\pi n$, then $\psi(\theta(\vec{r}) + 2\pi n) = |\psi| e^{i(\theta(\vec{r}) + 2\pi n)} = |\psi| e^{i\theta(\vec{r})} = \psi(\theta(\vec{r}))$, i.e. the wave function is uni-evaluated. It means that whatever the path is

$$\oint d\vec{s} \vec{\nabla} \theta = 2\pi n$$

should hold. We finally get

$$\Phi = n\Phi_0 = n \frac{h}{2e}$$

with $\Phi_0 = \frac{hc}{2e} = 2.07 \cdot 10^{-7} \text{ Gauss} \cdot \text{cm}^2$ (SGS) or $\Phi_0 = \frac{h}{2e} = 2.07 \cdot 10^{-15} \text{ Wb}$ (SI). The value Φ_0 is the superconducting flux quantum.

Case 2: macroscopic SC sample with a small circular hole.

As in the previous case we have a macroscopic superconductor, but now we consider a very small cylindrical hole. Let us consider a loop around it with a radius $R \ll \lambda_L$. At such a short distance from the edge there is no magnetic screening, and we can consider $\vec{B}(\vec{r}) = \mu_0 \vec{H}_{ext} = cte$.

Consider the second GL equation over the loop.

$$j_s = -i \frac{(2e)\hbar}{2m^*} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) - \frac{(2e)^2}{m^*} |\psi|^2 \vec{A}$$

We integrate et take into account $\vec{j}_s = en\vec{v}_s$; we get

$$\frac{m^*}{(2e)} \oint d\vec{s} \vec{v}_s = \frac{\hbar}{2e} \oint d\vec{s} \vec{\nabla} \theta - \oint d\vec{s} \vec{A}$$

The first term is the circulation of superconducting electrons; the second term is equal to $n\Phi_0$; the third term is the flux across the loop:

$$\Phi = n\Phi_0 - \frac{m^*}{(2e)} \oint d\vec{s} \vec{v}_s$$

i.e. for microscopic distances the flux is not quantized. To make it quantized we should require

$$\oint d\vec{s} \vec{v}_s = 0$$

NB. The circulation of the phase is always

$$\oint d\vec{s} \vec{\nabla} \theta = 2\pi n$$

Taking into account the circular symmetry of the problem we get: $\vec{\nabla} \theta = n/R$. This is peculiar: as if we make $R \rightarrow 0$ we get the phase singularity $\vec{\nabla} \theta \rightarrow \infty$. And it is indeed realized in superfluids/SCs – quantum vortices!

Case 3: macroscopic SC ring.

Let us now apply the GL theory to the case of a superconducting ring, as shown in Fig. 3.4. Describing the system using cylindrical polar coordinates, $\mathbf{r} = (r, \phi, z)$, with the z -axis perpendicular to the plane of the ring, we see that the order parameter $\psi(\mathbf{r})$ must be periodic in the angle ϕ ,

$$\psi(r, \phi, z) = \psi(r, \phi + 2\pi, z). \quad (4.68)$$

We assume that the variations of $\psi(\mathbf{r})$ across the cross section of the ring are unimportant, and so we can neglect r and z dependence. Therefore the possible order parameters inside the superconductor are of the form

$$\psi(\phi) = \psi_0 e^{in\phi}, \quad (4.69)$$

where n is an integer and ψ_0 is a constant. We can interpret n as a **winding number** of the macroscopic wave function, exactly as for the case of superfluid helium in Fig. 2.9.

However, unlike the case of superfluid helium, a circulating current in a superconductor will induce magnetic fields. Assuming that there is a magnetic flux Φ through the ring, then the vector potential can be chosen to be in the tangential direction, $\mathbf{e}_\phi$ and is given by

$$A_\phi = \frac{\Phi}{2\pi R}, \quad (4.70)$$

where R is the radius of the area enclosed by the ring. This follows from

$$\Phi \equiv \int \mathbf{B} \cdot d\mathbf{S} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint \mathbf{A} \cdot d\mathbf{r} = 2\pi R A_\phi. \quad (4.71)$$

The free energy corresponding to this wave function and vector potential is

$$\begin{aligned} F_s(T) &= F_n(T) + \int d^3r \left(\frac{\hbar^2}{2m^*} \left| \left(\nabla + \frac{2ei}{\hbar} \mathbf{A} \right) \psi \right|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 \right) + E_B \\ &= F_s^0(T) + V \left(\frac{\hbar^2}{2m^*} \left| \frac{in}{R} - \frac{2ei\Phi}{2\pi\hbar R} \right|^2 |\psi|^2 \right) + \frac{1}{2\mu_0} \int B^2 d^3r \end{aligned} \quad (4.72)$$

where we have used the expression to gradient in cylindrical polar coordinates

$$\nabla X = \frac{\partial X}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial X}{\partial \phi} \mathbf{e}_\phi + \frac{\partial X}{\partial z} \mathbf{e}_z \quad (4.73)$$

(Boas 1983), V is the total volume of the superconducting ring, and $F_s^0(T)$ is the ground state free energy of the ring in the absence of any currents and magnetic fluxes. The vacuum magnetic field energy $E_B = (1/2\mu_0) \int B^2 d^3r$ can be expressed in terms of the inductance, L of the ring and the current I ,

$$E_B = \frac{1}{2} L I^2. \quad (4.74)$$

Clearly, it will be proportional to the square of the total flux, Φ through the ring

$$E_B \propto \Phi^2.$$

On the other hand, the energy of the superconductor contains a term depending on both the flux Φ and the winding number, n . This term can be expressed as,

$$V \frac{\hbar^2}{2m^* R^2} |\psi|^2 (\Phi - n\Phi_0)^2,$$

where the **flux quantum** is $\Phi_0 = h/2e = 2.07 \times 10^{-15}$ Wb.

We therefore see that the free energy is equal to the bulk free energy plus two additional terms depending only on the winding number n and the flux Φ . The energy of the superconducting ring is therefore of the general form,

$$F_s(T) = F_s^{\text{bulk}}(T) + \text{const.}(\Phi - n\Phi_0)^2 + \text{const.}\Phi^2. \quad (4.75)$$

This energy is sketched in Fig. 4.6. We can see from the figure that the free energy is a minimum whenever the flux through the loop obeys $\Phi = n\Phi_0$. This is the phenomenon of **flux quantization** in superconductors.

Taking a ring in its normal state above T_c and cooling it to below T_c will result in the system adopting one of the metastable minima in Fig. 4.6, depending on the applied field. It will then be trapped in the minimum, and a persistent current will flow around the ring to maintain a constant flux $\Phi = n\Phi_0$. Even if any external magnetic fields are turned off, the current in the ring must maintain

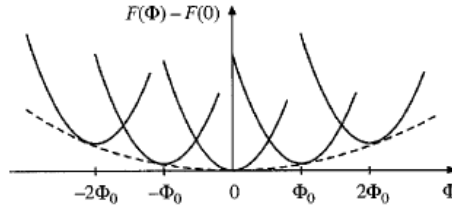


Fig. 4.6 Flux quantization in a superconducting ring. Metastable energy minima exist when the flux is an integer multiple of the flux quantum $\Phi_0 = h/2e$. There is an overall background increase with Φ corresponding to the self-inductance of the ring, making the zero flux state $\Phi = 0$ the global energy minimum. Thermal fluctuations and quantum tunnelling allow transitions between neighboring metastable energy minima.

a constant flux Φ in the ring. It is possible to directly measure the magnetic flux directly in such rings, and hence confirm that it is indeed quantized in units of Φ_0 , or multiples of 2×10^{-15} Wb. Incidentally, the fact that flux quantization is observed in units of $\Phi_0 = h/2e$ and not units of h/e is clear experimental proof that the relevant charge is $2e$ not e , hence implying the existence of Cooper pairs.

Given that a system is prepared in one of the metastable minima, it can, in principle, escape over the energy barriers to move into a neighboring lower energy minimum. This would be a mechanism for the persistent current to decay, and hence for dissipation. Such an event corresponds to a change in the winding number, n , and is called a **phase-slip**. However, the rate for thermally hopping over these barriers is exponentially small, of order

$$\frac{1}{\tau} \sim e^{-E_0/k_B T} \quad (4.76)$$

where E_0 is the barrier height between minima in Fig. 4.6. Clearly this thermal hopping rate can be made negligibly small. For example, E_0 is formally proportional to the ring volume, V , and so can be made arbitrarily large in a macroscopic system. In practice persistent currents have been observed to flow for years, with essentially no decay!

Another interesting possible mechanism for a phase-slip would be a quantum tunnelling from one minimum to another. This would be possible at any temperature. But again the rate is impractically small in macroscopic systems. However, a very interesting recent development has been the direct observation of these tunnelling events in small **mesoscopic** superconducting rings. These experiments have demonstrated **macroscopic quantum coherence** and are discussed briefly in the next chapter.

4.10. Gauge symmetry and symmetry breaking

The GL order parameter for superconductors has both an amplitude and a complex phase

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})|e^{i\theta(\mathbf{r})}. \quad (4.54)$$

This is similar to the macroscopic wave function for superfluid He II, introduced in Chapter 2. However, unlike superfluids of neutral particles, something very interesting happens now when we consider gauge invariance.

If we make a **gauge transformation** of the magnetic vector potential

$$\mathbf{A}(\mathbf{r}) \rightarrow \mathbf{A}(\mathbf{r}) + \nabla\chi(\mathbf{r}) \quad (4.55)$$

then we must make a corresponding change in the phase of the order parameter, θ . Consider the term in the GL free energy density containing the canonical momentum operator

$$\hat{p} = \frac{\hbar}{i}\nabla + 2e\mathbf{A}.$$

If we change the phase of the order parameter by

$$\psi(\mathbf{r}) \rightarrow \psi(\mathbf{r})e^{i\theta(\mathbf{r})} \quad (4.56)$$

then we obtain

$$\begin{aligned} \hat{p}\psi(\mathbf{r})e^{i\theta(\mathbf{r})} &= e^{i\theta(\mathbf{r})}\left(\frac{\hbar}{i}\nabla + 2e\mathbf{A}\right)\psi(\mathbf{r}) + \psi(\mathbf{r})e^{i\theta(\mathbf{r})}\hbar\nabla\theta(\mathbf{r}), \\ &= e^{i\theta(\mathbf{r})}\left(\frac{\hbar}{i}\nabla + 2e\left(\mathbf{A} + \frac{\hbar}{2e}\nabla\theta\right)\right)\psi(\mathbf{r}). \end{aligned} \quad (4.57)$$

From this it follows that the free energy will be unchanged when we simultaneously change $\psi(\mathbf{r})$ to $\psi(\mathbf{r})e^{i\theta(\mathbf{r})}$ and the vector potential according to

$$\mathbf{A}(\mathbf{r}) \rightarrow \mathbf{A}(\mathbf{r}) + \frac{\hbar}{2e}\nabla\theta. \quad (4.58)$$

This shows that the theory satisfies **local gauge invariance**. Both the phase of the order parameter and the magnetic vector potential depend on the choice of gauge, but all physical observables (free energy, magnetic field $\mathbf{B}$ etc.) are gauge invariant.

So far this is all completely general. But we saw earlier that a bulk superconductor has a ground state with a constant order parameter, ψ . Therefore it must have the same θ everywhere. There must be a **phase-stiffness**, or an energy cost associated with changing θ from one part of the solid to another. If we consider a superconductor in which the order parameter has a constant magnitude, $|\psi|$, and a phase $\theta(\mathbf{r})$, which varies only slowly with position $\mathbf{r}$, then (using Eq. 4.57) we obtain the total free energy

$$F_s = F_s^0 + \rho_s \int d^3r \left(\nabla\theta + \frac{2e}{\hbar}\mathbf{A} \right)^2. \quad (4.59)$$

Here the **superfluid stiffness** is defined by,

$$\rho_s = \frac{\hbar^2}{2m^*} |\psi|^2 \quad (4.60)$$

and F_s^0 is the total free energy in the ground state ($\theta = \text{constant}$, $\mathbf{A} = 0$). Now if we choose some particular gauge for $\mathbf{A}(\mathbf{r})$, such as the London gauge, $\nabla \cdot \mathbf{A} = 0$,

then within this fixed gauge there is now a free energy cost associated with further gradients in $\theta(\mathbf{r})$. To minimize the gradient energy, we must minimize the gradients, by making $\theta(\mathbf{r})$ as constant as possible throughout the system. In the case of zero applied magnetic field, we can choose $\mathbf{A} = 0$, and clearly then $\theta(\mathbf{r})$ will be constant everywhere in the system. Again we are back to the XY symmetry of Fig. 2.4. Since the system effectively chooses an (arbitrary) constant order parameter phase everywhere in the system, we can say that the system exhibits **long ranged order** in the order parameter phase, just as a ferromagnet has long ranged order in its magnetization $\mathbf{M}(\mathbf{r})$.

Because the long ranged order is in the phase variable (which is not normally a physical observable in quantum mechanics) we say that the system has **spontaneously broken global gauge symmetry**. The point is that *global* gauge symmetry refers to changing $\theta(\mathbf{r})$ by a constant amount everywhere in the whole solid (which does not require any change in $\mathbf{A}$). This is in contrast to *local* gauge symmetry in which $\theta(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ are changed simultaneously, consistent with Eq. 4.58.

Equation 4.59 also implies the London equation, and hence the Meissner–Ochsenfeld effect, bringing us full circle back to Chapter 3. The current can be calculated from a functional derivative of the free energy

$$\begin{aligned}\mathbf{j}_s &= -\frac{\partial F_s[\mathbf{A}]}{\partial \mathbf{A}(\mathbf{r})}, \\ &= -\frac{2e}{\hbar} \rho_s \left(\nabla \theta + \frac{2e}{\hbar} \mathbf{A} \right).\end{aligned}\quad (4.61)$$

Starting in the ground state, where θ is constant, we directly find that with a small constant external vector potential, $\mathbf{A}$, the current is,

$$\mathbf{j}_s = -\rho_s \frac{(2e)^2}{\hbar^2} \mathbf{A}, \quad (4.62)$$

which is exactly the same as the London equation. The superfluid stiffness, ρ_s , is essentially just the London superfluid density, n_s , in disguise!

To make the connection between ρ_s and the London superfluid fraction, n_s , more clear, consider the London equation

$$\mathbf{j}_s = -\frac{n_s e^2}{m_e} \mathbf{A}. \quad (4.63)$$

If we rewrite Eq. 4.61 in the form

$$\mathbf{j}_s = -\frac{(2e)^2}{2m^*} |\psi|^2 \mathbf{A}, \quad (4.64)$$

then clearly these are the same. It is conventional to define the constants so that the London superfluid density is $n_s = 2|\psi|^2$ and the GL effective mass is $m^* = 2m_e$ (where m_e is the bare electron mass). With this choice the equation can be interpreted physically as implying that $|\psi|^2$ is the density of pairs of electrons in the ground state. Therefore in comparison with the BCS theory of superconductivity we can interpret $|\psi|^2$ with the density of Cooper pairs in the ground state, and n_s as the density of electrons belonging to these Cooper pairs. The normal fraction, $n_n = n - n_s$ corresponds to the density of unpaired electrons. The GL parameter m^* is the mass of the Cooper pair, which is naturally twice the original electron mass.

Table 4.1 Penetration depth, $\lambda(0)$, and coherence length, $\xi(0)$, at zero temperature for some important superconductors. Data values are taken from Poole (2000).

	T_c (K)	$\lambda(0)$ (nm)	$\xi(0)$ (nm)	κ
Al	1.18	1550	45	0.03
Sn	3.72	180	42	0.23
Pb	7.20	87	39	0.48
Nb	9.25	39	52	1.3
Nb ₃ Ge	23.2	3	90	30
YNi ₂ B ₂ C	15	8.1	103	12.7
K ₃ C ₆₀	19.4	2.8	240	95
YBa ₂ Cu ₃ O _{7-δ}	91	1.65	156	95

In terms of the original free energy GL parameters, $\dot{a}$ and b , the superfluid density, n_s is given by

$$n_s = 2|\psi|^2 = 2 \frac{\dot{a}(T_c - T)}{b}. \quad (4.65)$$

Therefore the London penetration depth, $\lambda(T)$ is given by

$$\lambda(T) = \left(\frac{m_e b}{2\mu_0 e^2 \dot{a}(T_c - T)} \right)^{1/2}. \quad (4.66)$$

Clearly this will diverge at the critical temperature, T_c , since it is proportional to $(T_c - T)^{-1/2}$. We saw earlier that the GL coherence length, $\xi(T)$, also diverges with the same power of $(T_c - T)$, and so the dimensionless ratio,

$$\kappa = \frac{\lambda(T)}{\xi(T)}, \quad (4.67)$$

is independent of temperature within the GL theory. Table 4.1 summarizes the measured values of penetration depth and coherence length at zero temperature, $\lambda(0)$, $\xi(0)$, for a selection of superconductors.

Hour 8.

5. Vortex Matter in Superconductors

5.1. S-N interface energy. Type I and Type II SCs.

Here we will try to understand why there are two kinds (types) of superconductors. For this we will consider a SC sample at a field very close to the critical one. We will calculate the energy at the interface between already normal and yet superconducting regions of the sample.

First step. It is possible to express the thermodynamical critical field in terms of ξ and λ . We know that the total condensation energy is $F_s - F_N = -\frac{\mu_0 H_c^2}{2}$. Within the GL theory we showed that at equilibrium the OP ψ takes the value

$$|\psi|^2 = -\frac{\alpha}{\beta} = \frac{|\alpha|}{\beta}$$

At this this (thermodynamic equilibrium) value the free energy is

$$F_s = F_N - |\alpha||\psi|^2 + \frac{\beta}{2}|\psi|^4 = F_N - \frac{|\alpha|^2}{2\beta} = F_N - \frac{\tilde{a}^2}{2b}(T_c - T)^2$$

We therefore have :

$$H_c = \frac{\tilde{a}}{\sqrt{\mu_0 b}}(T_c - T)$$

We now replace $\tilde{a}$ and b by characteristic lengths of superconductivity :

$$\xi(T) = \left(\frac{\hbar^2}{2m^*|\tilde{a}(T_c - T)|} \right)^{1/2}$$

$$\lambda(T) = \left(\frac{m^*b}{4\mu_0 e^2|\tilde{a}(T_c - T)|} \right)^{1/2}$$

We notice that

$$\frac{1}{\xi(T)\lambda(T)} = \left(\frac{4\mu_0 e^2|\tilde{a}(T_c - T)|2m^*|\tilde{a}(T_c - T)|}{\hbar^2 m^* b} \right)^{1/2} = \frac{2e}{\hbar} \sqrt{2\mu_0} H_c$$

And we finally get for $H_c(T)$

$$\mu_0 H_c(T) = \frac{\Phi_0}{2\pi\sqrt{2}\lambda(T)\xi(T)}$$

Now it becomes possible to rewrite GL equations using the reduced OP $|\psi_r(\vec{r})|^2 \in [0,1]$, as

$$\psi(\vec{r}) = \psi_0 \psi_r(\vec{r}) \quad (\xi^2 = \frac{\hbar^2}{2m^*|\alpha|} \text{ and } \psi_0 = \sqrt{\frac{|\alpha|}{\beta}}):$$

$$-\xi^2 \left(\vec{\nabla} + \frac{i2e}{\hbar} \vec{A} \right)^2 \psi_r(\vec{r}) - \psi_r(\vec{r}) + |\psi_r(\vec{r})|^2 \psi_r(\vec{r}) = 0$$

In this reduced Op notation the free energy density:

$$f_s(T) = f_n(T) + \frac{\hbar^2}{2m^*} \left| \left(\vec{\nabla} + \frac{i2e}{\hbar} \vec{A} \right) \psi_r(\vec{r}) \right|^2 - |\alpha||\psi_r(\vec{r})|^2 + \frac{\beta}{2} |\psi_r(\vec{r})|^4$$

becomes for the reduced OP $\psi(\vec{r}) \rightarrow \psi_r(\vec{r}) \sqrt{\frac{|\alpha|}{\beta}}$

$$f_s(T) = f_n(T) + \frac{|\alpha|}{\beta} \frac{\hbar^2}{2m^*} \left| \left(\vec{\nabla} + \frac{i2e}{\hbar} \vec{A} \right) \psi_r(\vec{r}) \right|^2 - \frac{|\alpha|}{\beta} |\alpha| |\psi_r(\vec{r})|^2 + \left(\frac{|\alpha|}{\beta} \right)^2 \frac{\beta}{2} |\psi_r(\vec{r})|^4$$

$$f_s(T) = f_n(T) + \frac{|\alpha|^2}{\beta} \xi^2 \left| \left(\vec{\nabla} + \frac{i2e}{\hbar} \vec{A} \right) \psi_r(\vec{r}) \right|^2 - \frac{|\alpha|}{\beta} |\alpha| |\psi_r(\vec{r})|^2 + \left(\frac{|\alpha|}{\beta} \right)^2 \frac{\beta}{2} |\psi_r(\vec{r})|^4$$

Now, remember that the condensation energy is $-\frac{|\alpha|^2}{2\beta} = -\frac{\mu_0 H_c^2}{2}$. We can write :

$$f_s(T) = f_n(T) + 2 \frac{\mu_0 H_c^2}{2} \left[-\xi^2 \left| \left(\vec{\nabla} + \frac{i2e}{\hbar} \vec{A} \right) \psi_r(\vec{r}) \right|^2 - |\psi_r(\vec{r})|^2 + \frac{1}{2} |\psi_r(\vec{r})|^4 \right]$$

The total free energy should also include the energy of the expelled field $\frac{\mu_0 H^2}{2}$:

$$f_s(T, H) = f_n(T) + 2 \frac{\mu_0 H_c^2}{2} \left[-\xi^2 \left| \left(\vec{\nabla} + \frac{i2e}{\hbar} \vec{A} \right) \psi_r(\vec{r}) \right|^2 - |\psi_r(\vec{r})|^2 + \frac{1}{2} |\psi_r(\vec{r})|^4 \right] + \frac{\mu_0 H^2}{2}$$

One can now calculate the Gibbs energy in terms of $\psi_r(\vec{r})$.

Second step. Let us now consider the SC in a magnetic field very close to the critical one. If the sample is a long cylinder – we know the solution. However, if the shape of the sample is different, the magnetic field will penetrate first at its sides where the magnetic energy is higher. Close to transition $H \leq H_c$ it will lead to a non-homogeneous intermediate state of a SC where normal and superconducting regions alternate. In fact, we already showed that the Gibbs energy of the normal part (with penetrated critical field $\mu_0 H_c$ is exactly equal to the Gibbs energy of superconducting parts where $\mu_0 H = 0$. The only energy term which may remain and may decide the existence of the intermediate state is the energy near S-N boundaries. Indeed, at the S-N interface two things happen : i) – the field is screened in S-part but penetrates the N-part on the scale of $\lambda(T)$; ii) –the OP is maximum far in the S-part and vanishes deep in the N-part on a scale of $\xi(T)$. Since both OP and magnetic field enter the free energy, the free energy near the interface could be positive or negative depending on $\kappa = \lambda/\xi$.

Third Step. Consider a S-N boundary perpendicular to x-axis ; the magnetic field is parallel to the boundary $\vec{H} = (0, 0, H(x))$. If one takes Landau gauge $\vec{A} = (0, A(x), 0)$ one obtain GL equations in the form:

$$-\xi^2 \frac{d^2}{dx^2} \psi_r(x) + \xi^2 \left(\frac{2\pi 2eA}{h} \right)^2 \psi_r(x) - \psi_r(x) + \psi_r(x)^3 = 0$$

$$\frac{d^2 A}{dx^2} = \left(\frac{\psi_r^2(x)}{\lambda^2} \right) A$$

By integrating one gets:

$$\left(1 - \frac{2\pi 2e\xi A}{h}\right)^2 \psi_r^2(x) - \frac{1}{2} \psi_r^4(x) + \left(\frac{2\pi 2e\xi \lambda}{h}\right)^2 \left(\frac{dA}{dx}\right)^2 + \xi^2 \left(\frac{d\psi_r(x)}{dx}\right)^2 = C$$

The constant $C=1/2$ is obtained from the condition in S-side far from the interface :

$$\psi_r(x \rightarrow -\infty) = 1, \frac{d\psi(x \rightarrow -\infty)}{dx} = 0, A(x \rightarrow -\infty) = 0$$

From this, and using the above obtained formula for critical field we obtain:

$$\left(\frac{2\pi 2e\xi A}{h} - 1\right)^2 \psi_r^2(x) + \frac{1}{2} \psi_r^4(x) = +\xi^2 \left(\frac{d\psi_r(x)}{dx}\right)^2 + \frac{B^2}{2\mu_0^2 H_c^2} - \frac{1}{2}$$

Fourth step. In the general case we defined the Gibbs energy density as $g = f - \mu_0 \vec{M} \vec{H}$, and

$$G_S - G_N = \int_V d^3r \left\{ \alpha |\psi(\vec{r})|^2 + \frac{\beta}{2} |\psi(\vec{r})|^4 + \frac{1}{2m^*} \left| \left(\frac{\hbar}{i} \vec{\nabla} - 2e\vec{A}(\vec{r}) \right) \psi(\vec{r}) \right|^2 \right\} \\ + \int_{V'} d^3r \left\{ \frac{\mu_0 H^2}{2} - \int_0^H \vec{B} d\vec{H} \right\}$$

In our case we need to integrate over the x-axis only. By putting the obtained above differential equation for $\psi_r(x)(B)$ we get:

$$\sigma_{ns}(T, H) = \int_{-\infty}^{+\infty} [G_s - G_n] dx = 4 \frac{\mu_0 H_c^2}{2} \int_{-\infty}^{+\infty} dx \left[\xi^2 \left(\frac{d\psi_r(x)}{dx} \right)^2 + \frac{B(B - \mu_0 H_c)}{2\mu_0^2 H_c^2} \right]$$

Let us analyze this result. At the S-N interface $\psi_r(x)$ in the first term varies between 1 and 0 at the scale of ξ . Therefore, the first integral is $\sim \xi$. The second term in the integral is always negative since $B \leq \mu_0 H_c$. Moreover, the magnetic term starts at 0 (since $B = 0$ deep in the superconducting part) and finishes at 0 (since $B = \mu_0 H_c$ in the normal part). This function is about 1 at its maximum, and it is non-zero at the scale of the penetration depth λ . Therefore, the second term is $\sim \lambda$. We immediately see that if

$$\kappa = \lambda / \xi \ll 1$$

then the interface energy is positive and the system will try to minimize the total surface of S/N boundaries. **This is the case in type I superconductors.** (NB : remind $M(H)$ plot here !) However, if

$$\kappa = \lambda / \xi \gg 1$$

Then the interface energy is negative

$$\sigma_{ns}(T, H) < 0$$

In this case the thermodynamic systems gains energy by multiplying S/N boundaries !!! **This is the case of type II superconductors.** At the end the type II SCs create a large number of S-N boundaries. One sees that the system, still superconducting, will « prefer » to have a huge number of S/N interfaces. In this case each N-region will be as small as a SC can admit. Moreover, each such N-region should be cylinder-like (the largest surface with the minimum volume). The minimum spatial scale that SC OP allows is ξ - the minimum possible length in this case! This is the case of Abrikosov vortex. Abrikosov showed that the crossover between type I and type II SC takes place at

$$\kappa = \lambda/\xi = \frac{1}{\sqrt{2}}$$

5.2. Minimum size of the normal region. Abrikosov vortex.

We can ask ourselves what is the minimal N-region allowed in a type II SC; the answer is straightforward from our previous calculation of the typical size of inhomogeneous OP – it should be of the order of ξ . If we consider such a normal region surrounded by SC one, then the external magnetic field penetrates a “hole” over $> \xi$. Let us consider the flux over a closed loop chosen in S far enough ($\gg \lambda_L$) from the S/N interface. Consider 2nd GL equation:

$$j_s = -i \frac{(2e)\hbar}{2m^*} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) - \frac{(2e)^2}{m^*} |\psi|^2 \vec{A}$$

Since the loop is far from the hole we can consider the constant amplitude of the order parameter ($|\psi|$ does not depend on position), only the phase could depend on $\vec{r}$: $\psi(\theta(\vec{r})) = |\psi|e^{i\theta(\vec{r})}$. Putting this expression in the 2nd GL equation we obtain

$$j_s = \frac{(2e)^2}{m^*} |\psi|^2 \left[\frac{\hbar}{2e} \vec{\nabla} \theta - \vec{A} \right]$$

Since on the loop $j_s = 0$, it results in $\frac{\hbar}{2e} \vec{\nabla} \theta = \vec{A}$.

Let us take now the path integral over the loop from both parts of this equation. We get

$$\begin{aligned} \frac{\hbar}{2e} \oint d\vec{s} \vec{\nabla} \theta &= \oint d\vec{s} \vec{A} \\ \oint d\vec{s} \vec{A} &= \int d\vec{S} \text{rot}(\vec{A}) = \int d\vec{S} \vec{B} = \Phi \end{aligned}$$

Φ being the magnetic flux across the loop. Thus we have

$$\frac{\hbar}{2e} \oint d\vec{s} \vec{\nabla} \theta = \Phi$$

We already obtained this result in the previous lecture when considering flux quantization over a macroscopic hole in a SC specimen. Since $\psi(\vec{r})$ is uni-defined, the phase accumulation along the loop should be $2\pi n$,

$$\oint d\vec{s} \vec{\nabla} \theta = 2\pi n$$

In order to minimize the energy of currents circulating around “hole”, the value n is usually 1.

Notice that the integral

$$\oint d\vec{s} \vec{\nabla} \theta = \int d\vec{S} \text{rot}(\vec{\nabla} \theta) = 2\pi$$

may be seen as $\text{rot}(\vec{\nabla} \theta) = 2\pi \delta(\vec{r}_0) \vec{u}_z$

i.e. we have a phase singularity in $\vec{r}_0$!! Indeed, taking into account the circular symmetry of the problem we get: $\vec{\nabla} \theta = 1/r$. This is peculiar: as if we make $r \rightarrow 0$ we get the phase singularity $\vec{\nabla} \theta \rightarrow \infty$. And it is indeed realized in superfluids/SCs – quantum vortices!

NB. The dimensionality of $\delta(\vec{r}_0) = [m^{-2}]$

5.3. Abrikosov vortex lattice (optional)

The great beauty of the GL theory is that it allows one to solve many difficult problems in superconductivity, without any reference to the underlying microscopic BCS theory. In some sense one could argue that it is more general, for example, it would almost certainly apply to exotic superconductors, such as the high T_c cuprates, even though the original BCS theory does not seem to explain these systems. The other great advantage of the GL theory is that it is considerably easier to work with than the BCS theory, especially in cases

where the order parameter has complicated spatial variations. The *tour de force* example of this is the **Abrikosov flux lattice**.

Abrikosov found a solution to the GL equations in the case of a bulk superconductor in a magnetic field.⁷ The result he obtained is remarkable in many respects. It is essentially an exact solution for type II superconductors, valid close to H_{c2} . Furthermore, it predicted a striking result, that just below H_{c2} the order parameter forms into a periodic structure of vortices. Each vortex carries a magnetic flux, hence explaining how magnetic flux enters superconductors in the **mixed state** between H_{c1} and H_{c2} . Abrikosov's prediction of the flux lattice was confirmed experimentally, showing not just that vortices occur, but they also tend to align in a regular triangular lattice, as predicted by the theory. This periodic lattice of vortices was perhaps the first example in physics of *emergent phenomena in complex systems*; the fact that sufficiently complex systems exhibit a variety of novel phenomena on different length scales. These phenomena effectively arise from *self-organization* on the macroscopic length scale.

In type II superconductors the thermodynamic phase transition at H_{c2} is second-order (see Exercise 4.1). Therefore we can expect that the GL order parameter ψ is small in magnitude just below H_{c2} and reaches zero exactly at H_{c2} .⁸ Therefore, the magnetization M will also be small at a magnetic field just below H_{c2} (since ψ is near zero, the superfluid density n_s and the screening supercurrents will also vanish at H_{c2}), as can be seen in Fig. 3.8. Therefore to a good approximation we can assume that

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad (4.77)$$

where $\mathbf{H}$ is, as usual, the applied field given by the external apparatus as in Fig. 3.6. This also implies that sufficiently near to H_{c2} we can neglect any spatial variations in the B-field, $\mathbf{B}(\mathbf{r})$ and just treat it as a constant,

$$\mathbf{B} = (0, 0, B). \quad (4.78)$$

It will be convenient to express the corresponding vector potential $\mathbf{A}$ in the **Landau gauge** as

$$\mathbf{A}(\mathbf{r}) = (0, xB, 0). \quad (4.79)$$

In which case the GL equation, Eq. 4.50 becomes

$$-\frac{\hbar^2}{2m^*} \left(\nabla + \frac{2eBi}{\hbar} x \mathbf{e}_y \right) \cdot \left(\nabla + \frac{2eBi}{\hbar} x \mathbf{e}_y \right) \psi(\mathbf{r}) + a(T)\psi + b|\psi|^2\psi = 0, \quad (4.80)$$

where, as usual, $\mathbf{e}_y$ is the unit vector in the y direction.

Now if we are infinitesimally below H_{c2} , then ψ is essentially zero and we can drop the cubic term, $b|\psi|^2\psi$. All the other terms are linear in ψ and so we have **linearized** the equation. Expanding out the bracket (paying attention to the commutation of ∇ and $x\mathbf{e}_y$) gives

$$-\frac{\hbar^2}{2m^*} \left(\nabla^2 + \frac{4eBi}{\hbar} x \frac{\partial}{\partial y} - \frac{(2eB)^2}{\hbar^2} x^2 \right) \psi(\mathbf{r}) + a(T)\psi = 0. \quad (4.81)$$

Introducing the cyclotron frequency,

$$\omega_c = \frac{2eB}{m^*}, \quad (4.82)$$

⁷Abrikosov was awarded the 2003 Nobel Prize for Physics for this work. He shared the prize with Ginzburg and Leggett.

⁸In contrast this would not be the case for a type I superconductor, where the transition at H_c is first order. In these superconductors ψ jumps from discontinuously from zero to a finite value at H_c .

and noting that $a(T)$ is negative since we are at a temperature below the zero field T_c , the equation can be written in the form,

$$\left(-\frac{\hbar^2}{2m^*}\nabla^2 - \hbar\omega_c ix \frac{\partial}{\partial y} + \frac{m^*\omega_c^2}{2}x^2\right)\psi(\mathbf{r}) = |a|\psi(\mathbf{r}). \quad (4.83)$$

where $\xi(T)$ is the GL coherence length.

Now Eq. 4.83 has the form of an eigenvalue equation, and is well known in quantum mechanics. It is equivalent to the Schrödinger equation for the wave function of a charged particle in a magnetic field, which has well known **Landau level** solutions (Ziman 1979). The solution has the form,

$$\psi(\mathbf{r}) = e^{i(k_y y + k_z z)} f(x), \quad (4.84)$$

which is a combination of plane waves in the y and z directions, combined with an unknown function of x , $f(x)$.

To find an equation for this function, $f(x)$ we substitute the trial solution into Eq. 4.83. We find that $f(x)$ obeys

$$-\frac{\hbar^2}{2m^*}\frac{d^2f}{dx^2} + \left(\hbar\omega_c k_y x + \frac{m^*\omega_c^2}{2}x^2\right)f = \left(|a| - \frac{\hbar^2(k_y^2 + k_z^2)}{2m^*}\right)f. \quad (4.85)$$

The term in brackets on the left hand side can be rearranged by “completing the square,”

$$\left(\hbar\omega_c k_y x + \frac{m^*\omega_c^2}{2}x^2\right) = \frac{m^*\omega_c^2}{2}(x - x_0)^2 - \frac{m^*\omega_c^2}{2}x_0^2, \quad (4.86)$$

where

$$x_0 = -\frac{\hbar k_y}{m\omega_c}. \quad (4.87)$$

Finally, moving all the constants over to the right-hand-side we find,

$$-\frac{\hbar^2}{2m^*}\frac{d^2f}{dx^2} + \frac{m^*\omega_c^2}{2}(x - x_0)^2 f = \left(|a| - \frac{\hbar^2 k_z^2}{2m^*}\right)f. \quad (4.88)$$

Equation 4.88 is just the Schrödinger equation for a simple harmonic oscillator, except that the origin of coordinates is shifted from $x = 0$ to $x = x_0$. Therefore the term in brackets on the right is just the energy of the oscillator,

$$\left(n + \frac{1}{2}\right)\hbar\omega_c = |a| - \frac{\hbar^2 k_z^2}{2m^*}, \quad (4.89)$$

or

$$\left(n + \frac{1}{2}\right)\hbar\omega_c + \frac{\hbar^2 k_z^2}{2m^*} = a(T_c - T). \quad (4.90)$$

The corresponding functions $f(x)$ are just the wave functions of a simple Harmonic oscillator for each n , shifted by x_0 .

Imagine that we gradually cool a superconductor in an external field, H . At the zero field transition temperature, T_c , it will be impossible to satisfy Eq. 4.90 because of the zero point energy term $\hbar\omega_c/2$ on the left hand side. A solution will only be possible when the temperature is far enough below T_c to achieve,

$$\frac{1}{2}\hbar\omega_c = a(T_c - T), \quad (4.91)$$

corresponding to the lowest possible energy solution ($n = 0$, $k_z = 0$). This equation determines the depression in transition temperature in the magnetic

field,

$$\begin{aligned} T_c(H) &= T_c(0) - \frac{1}{2\dot{a}} \hbar \omega_c, \\ &= T_c(0) - \frac{2e\hbar\mu_0}{2\dot{a}m^*} H. \end{aligned} \quad (4.92)$$

Alternatively, we can start in a large external field, H , above H_{c2} which we gradually decrease (keeping the temperature fixed) until

$$\frac{1}{2} \hbar \frac{2eB}{m^*} = \dot{a}(T_c - T). \quad (4.93)$$

Therefore, rearranging,

$$\begin{aligned} \mu_0 H_{c2} = B_{c2} &= \frac{2m^*\dot{a}(T_c - T)}{\hbar^2} \frac{\hbar}{2e}, \\ &= \frac{\Phi_0}{2\pi\xi(T)^2}. \end{aligned} \quad (4.94)$$

It is interesting to note that this result implies that at H_{c2} there is exactly one flux quantum (i.e. one vortex line), in each unit area $2\pi\xi(T)^2$. This expression also provides the simplest way to measure the GL coherence length $\xi(0)$ experimentally. Since $\xi(T) = \xi(0)t^{-1/2}$ where $t = |T - T_c|/T_c$,

$$\mu_0 H_{c2} = \frac{\Phi_0}{2\pi\xi(0)^2} \frac{T_c - T}{T_c}, \quad (4.95)$$

and so by measuring the gradient of $H_{c2}(T)$ near to T_c one can easily find the corresponding $\xi(0)$.

It is also interesting to compare this expression for H_{c2} with the corresponding result for H_c which we found earlier. Using the GL expression for the total energy in the Meissner state we had found that

$$\begin{aligned} H_c &= \frac{\dot{a}}{(\mu_0 b)^{1/2}} (T_c - T), \\ &= \frac{\Phi_0}{2\pi\mu_0\sqrt{2}\xi\lambda}, \\ &= \frac{H_{c2}}{\sqrt{2}\kappa}, \end{aligned} \quad (4.96)$$

and therefore

$$H_{c2} = \sqrt{2}\kappa H_c. \quad (4.97)$$

From this we can deduce that for superconductors with $\kappa > 1/\sqrt{2}$ we will have $H_{c2} > H_c$, and the phase transition will be second-order with the order parameter growing continuously from zero at fields just below H_{c2} , that is, we will have a type II superconductor. On the other hand, for superconductors with $\kappa < 1/\sqrt{2}$ we will have $H_{c2} < H_c$, and the phase transition will be a first-order phase transition at the field H_c , below which the order parameter jumps discontinuously to a finite value. The Abrikosov theory therefore describes the difference between type I and type II superconductors,

$$\kappa \begin{cases} < \frac{1}{\sqrt{2}} & \text{type I,} \\ > \frac{1}{\sqrt{2}} & \text{type II.} \end{cases}$$

The linearized GL equation allows us to find at H_{c2} , but does not immediately tell us anything about the form of the solution below this field. To do this we

must solve the nonlinear equation, Eq. 4.80. This is very hard to do in general, but Abrikosov made a brilliant guess and came up with essentially the exact solution! He could see from the solutions to the linearized equation, Eq. 4.80, that only the harmonic oscillator ground state solutions $n = 0$ and $k_z = 0$ will be significant. However, there are still an infinite number of degenerate states, corresponding to the different possible k_y values,

$$\psi(\mathbf{r}) = C e^{i(k_y y)} e^{-(x-x_0)^2/\xi(T)^2}, \quad (4.98)$$

where C is a normalization constant. Here we have used the fact that the ground state wave function of a quantum harmonic oscillator is a gaussian function. The width of the gaussian solution to Eq. 4.88 turns out to be the GL coherence length $\xi(T)$.

Abrikosov's trial solution was to assume that we can combine these solutions into a **periodic** lattice. If we look for a solution which is periodic in y with period l_y , then we can restrict the values of k_y to

$$k_y = \frac{2\pi}{l_y} n \quad (4.99)$$

with n any positive or negative integer. The corresponding Landau level x shift is

$$x_0 = -\frac{2\pi\hbar}{m\omega_c l_y} n = -\frac{\Phi_0}{Bl_y} n. \quad (4.100)$$

Therefore we can try a periodic solution

$$\psi(\mathbf{r}) = \sum_{n=-\infty, \infty} C_n e^{i(2\pi n y/l_y)} e^{-(x+n\Phi_0/Bl_y)^2/\xi(T)^2}. \quad (4.101)$$

In this solution we can view the parameters C_n as variational parameters which are to be chosen to minimize the total GL free energy of the system.

The solution above is periodic in y , but not necessarily periodic in x . Abrikosov noted that it can be made periodic in x provided the coefficients obey

$$C_{n+\nu} = C_n \quad (4.102)$$

for some integer ν . The period is l_x , where

$$l_x = \nu \frac{\Phi_0}{Bl_y}. \quad (4.103)$$

Abrikosov studied the simplest case, $\nu = 1$ which corresponds to a simple square lattice. Later it was shown that a slightly lower total energy is obtained for the $\nu = 2$ case and the minimum energy state corresponds to a simple triangular lattice. In each case the order parameter $\psi(\mathbf{r})$ goes to zero at one point in each unit cell, and there is exactly one flux quantum Φ_0 per unit cell. Therefore the solution is a **periodic lattice of vortices**. This is illustrated in Fig. 4.7.

Experimental evidence for the Abrikosov flux lattice comes from a variety of methods. In a "flux decoration" experiment small paramagnetic particles are dusted onto the surface of the superconductor (just like the children's experiment to see magnetic fields of a bar magnet using iron filings on paper above the magnet!). The particles concentrate at the points of highest magnetic field, that is, the vortices. Other similar methods involve scanning a small SQUID loop or Hall probe just above the surface of the superconductor to directly

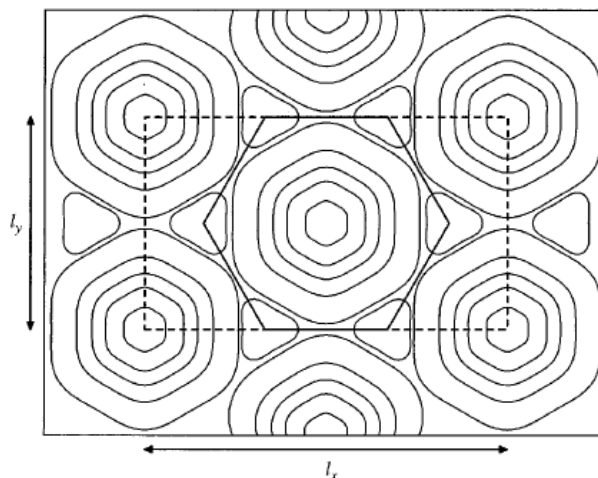


Fig. 4.7 The Abrikosov flux lattice. The figure shows the amplitude of the order parameter, $|\psi(\mathbf{r})|^2$ for the lowest energy triangular lattice. Each triangular unit cell contains one quantum, Φ_0 , of magnetic flux and contains one vortex where $\psi(\mathbf{r}) = 0$. In terms of the l_x and l_y lattice periodicities used in Section 4.9, $l_y = \sqrt{3}l_x$, and the rectangular unit cell $l_x \times l_y$ (dashed line) contains two vortices and two flux quanta.

measure the variation of the local flux density $\mathbf{B}(\mathbf{r})$. For most ordinary type II low T_c superconductors, such as Pb or Nb, these experiments indeed show that the vortices form a fairly regular hexagonal lattice. The lattice can be periodic over quite long length scales, but is disrupted now and then by defects. These defects (exactly analogous to crystal dislocations or point defects) tend to concentrate near defects in the underlying crystal lattice (such as grain boundaries, impurities, etc.).

Another method of observing the order in the flux lattice is to use neutron scattering. The neutrons have a magnetic moment, and so are sensitive to the magnetic field $\mathbf{B}(\mathbf{r})$. If this is periodic, as in the Abrikosov flux lattice, then there will be diffraction. The diffraction pattern can be used to find the geometry of the flux lattice. Again the majority of systems studied are found to have triangular lattices. But, interestingly, a square lattice has now been found in a few recently discovered superconductors. These are: the “borocarbide” system $\text{ErNi}_2\text{B}_2\text{C}$, the high temperature superconductor $\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$ and the possible p-wave superconductor strontium ruthenate Sr_2RuO_4 . All of these have been found to have square vortex lattices in at least some range of external fields. It may be that this is simply because of small corrections to the original Abrikosov theory (such as terms omitted in the standard GL equations, such as higher powers of ψ , like $|\psi|^6$, or higher order gradients). In fact the square lattice and triangular lattice have energies differing by less than 1% in the Abrikosov solution. But in some cases it seems more likely to be due to the fact that the underlying form of superconductivity is “unconventional,” meaning that the Cooper pairs have a different symmetry from the normal BCS case. We shall introduce these ideas briefly in Chapter 7.

Finally, we should note that while the Abrikosov solution is essentially exact just below H_{c2} , it cannot necessarily be applied far away from there, such as at H_{c1} . As we have seen, near to H_{c2} the vortices are close together, separated by distances of order the coherence length $\xi(T)$. Effectively they are so densely packed that their cores are essentially touching. On the other hand, just at H_{c1} we have very few vortices in the entire sample, and so they are well separated.

We can estimate of the lower critical field H_{c1} from the energy balance for the very first few vortices to enter a superconductor in the Meissner phase. One can show that a single London vortex has an energy of approximately (see Exercise 3.3)

$$E = \frac{\Phi_0^2}{4\pi\mu_0\lambda^2} \ln\left(\frac{\lambda}{\xi}\right) \quad (4.104)$$

per unit length. Therefore in a superconductor with N/A flux lines per unit area and thickness L , there will be a total energy cost EN/A per unit volume due to the vortices. But on the other hand, each vortex carries a flux Φ_0 , and so the average magnetic induction in the sample is $B = \Phi_0 N/A$. The magnetic work gained by the presence of the vortices is $\mu_0 H dM = H dB$ (at constant H). Energy balance therefore favors the presence of the vortices when

$$E \frac{N}{A} < H \Phi_0 \frac{N}{A}. \quad (4.105)$$

Thus it becomes energetically favorable for the vortices to enter the sample when $H > H_{c1}$, where

$$H_{c1} = \frac{\Phi_0}{4\pi\mu_0\lambda^2} \ln\left(\frac{\lambda}{\xi}\right). \quad (4.106)$$

This is obviously the lower critical field, and can be simply expressed as

$$H_{c1} = \frac{H_c}{\sqrt{2}\kappa} \ln(\kappa). \quad (4.107)$$

This expression is only valid when $\kappa \gg 1/\sqrt{2}$, that is, in the London vortex limit.