

## Simple Model for the Vortex Core in a Type II Superconductor

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*In order to model the core of an isolated vortex in a type II superconductor, a variational trial function for the magnitude of the normalized order parameter of the form  $f = \rho/R$  is assumed, where  $\rho$  is the radial coordinate,  $R = (\rho^2 + \xi_v^2)^{1/2}$ , and  $\xi_v$  is a variational core radius parameter. Remarkably simple analytic expressions for the magnetic flux density and supercurrent density that solve Ampere's law and the second Ginzburg-Landau equation are obtained. An analytic result for the free energy of the isolated vortex is then derived by integrating the Ginzburg-Landau free energy functional. The value of  $\xi_v$  that minimizes the free energy is calculated as a function of the Ginzburg-Landau parameter  $\kappa = \lambda/\xi$  and is found to range from  $\xi_v = 0.935\xi$  for  $\kappa = 0.707$  to  $\xi_v \approx 1.414\xi$  for  $\kappa \gg 1$ . A simple expression for the form factor or the Fourier transform of the flux density is obtained, which may be useful in the analysis of neutron diffraction experiments.*

The London model<sup>1,2</sup> of an isolated, singly quantized vortex line provides a useful tool for the phenomenological description of type II superconductors for which the Ginzburg-Landau parameter  $\kappa = \lambda/\xi$  obeys  $\kappa \gg 1$  (here  $\lambda$  is the penetration depth and  $\xi$  the coherence length). Although simple expressions for both the magnetic flux density and the supercurrent density can be calculated from this model, they suffer from the unrealistic property that they both diverge on the axis of the vortex line, because the depression of the order parameter to zero on the axis is not accounted for by the model.

We present in this paper some results that follow from the assumption of a simple variational model for the magnitude of the normalized order parameter of an isolated vortex:  $f = \rho/R$ , where  $\rho$  is the radial coordinate,  $R = (\rho^2 + \xi_v^2)^{1/2}$ , and  $\xi_v$  is a variational core radius parameter. Simple expressions for the magnetic flux density and the supercurrent density are

found, which reduce to the London model results outside the vortex core but have the added advantage of yielding realistic values in the vortex core vicinity. In addition, we obtain a convenient expression for the free energy of an isolated vortex, which involves modified Bessel functions. The value of  $\xi_v$  that appears in these expressions is easily calculated variationally and found to be of the order of  $\xi$ , the coherence length of the Ginzburg–Landau theory.<sup>3</sup>

Let us consider an isolated vortex line centered on the  $z$  axis. In terms of cylindrical coordinates  $\rho = (x^2 + y^2)^{1/2}$ ,  $\varphi = \tan^{-1}(y/x)$ , and  $z$ , with unit vectors  $\hat{\rho} = \hat{x} \cos \varphi + \hat{y} \sin \varphi$ ,  $\hat{\varphi} = \hat{y} \cos \varphi - \hat{x} \sin \varphi$ , and  $\hat{z}$ , the magnetic flux density is of the form  $\mathbf{b} = \hat{z} b_z(\rho)$ , and the supercurrent density can be written as  $\mathbf{j} = \hat{\varphi} j_\varphi(\rho)$ . We introduce a vector potential  $\mathbf{a}$  satisfying the Coulomb gauge,  $\nabla \cdot \mathbf{a} = 0$ , and the requirement that  $\mathbf{a}$  vanish on the vortex axis. If the normalized order parameter is of the form  $\Psi(\rho) = f(\rho) \exp(-i\varphi)$ , where  $f(\rho) = \rho/R = \rho/(\rho^2 + \xi_v^2)^{1/2}$ , the vector potential is of the form  $\mathbf{a} = \hat{\varphi} a_\varphi(\rho)$ , and the second Ginzburg–Landau equation can be written as

$$j_\varphi = -(c/4\pi\lambda^2)(a_\varphi - \varphi_0/2\pi\rho)f^2 \quad (1)$$

where  $\varphi_0 = hc/2e = 2.07 \times 10^{-7}$  G-cm<sup>2</sup> and  $\lambda$  is the penetration depth. With the help of  $\mathbf{b} = \nabla \times \mathbf{a}$  and Ampere's law,  $\mathbf{j} = (c/4\pi) \nabla \times \mathbf{b}$ , we obtain an inhomogeneous equation for  $a_\varphi$ :

$$\frac{d}{d\rho} \left[ \frac{1}{\rho} \frac{d}{d\rho} (\rho a_\varphi) \right] - \frac{f^2}{\lambda^2} a_\varphi = -\frac{\varphi_0 f^2}{2\pi\lambda^2 \rho} \quad (2)$$

whose solution is

$$a_\varphi = \frac{\varphi_0}{2\pi\rho} \left[ 1 - \frac{RK_1(R/\lambda)}{\xi_v K_1(\xi_v/\lambda)} \right] \quad (3)$$

where  $K_n(x)$  is a modified Bessel function. The corresponding magnetic flux density and supercurrent density are

$$b_z = (\varphi_0/2\pi\lambda\xi_v) K_0(R/\lambda)/K_1(\xi_v/\lambda) \quad (4)$$

$$j_\varphi = (c\varphi_0/8\pi^2\lambda^2\xi_v)(\rho/R)K_1(R/\lambda)/K_1(\xi_v/\lambda) \quad (5)$$

The  $z$  component of the flux through a circular area of radius  $\rho$  centered on the  $z$  axis is  $\Phi_z(\rho) = 2\pi\rho a_\varphi(\rho)$ . Figure 1 shows plots of  $b_z(\rho)/b_z(0)$ ,  $j_\varphi(\rho)/j_\varphi(\xi_v)$ ,  $\Phi_z(\rho)/\varphi_0$ , and  $f = \rho/(\rho^2 + \xi_v^2)^{1/2}$  vs.  $\rho/\xi_v$  for  $\lambda/\xi_v = 4$ . Note that these quantities have the same qualitative behavior as found by solving the Ginzburg–Landau equations numerically,<sup>4</sup> but were calculated in a matter of seconds with the help of a short computer program.

As can be shown from Eq. (13) of Ref. 1, the Ginzburg–Landau free energy density  $F$  associated with a vortex, measured relative to the free

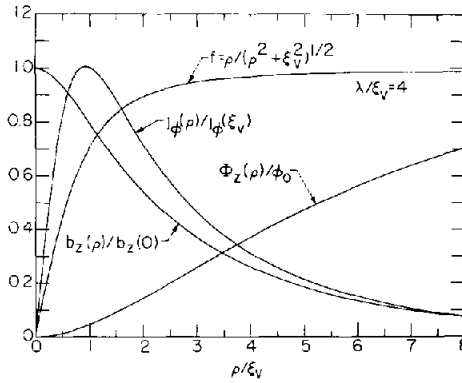


Fig. 1. Reduced magnetic flux density  $b_z(\rho)/b_z(0)$ , supercurrent density  $j_s(\rho)/j_s(\xi_v)$ , flux  $\Phi_s(\rho)/\Phi_0$ , and order parameter magnitude  $f = \rho/(\rho^2 + \xi_v^2)^{1/2}$  vs.  $\rho/\xi_v$ , where  $\rho$  is the radial coordinate and  $\xi_v$  is the variational core radius parameter, calculated from Eqs. (3)–(5) for  $\lambda/\xi_v = 4$ .

energy of the Meissner state, can be expressed in dimensionless form as

$$F' = 4\pi F/H_c^2 = F'_c + F'_k + F'_f \quad (6)$$

where

$$F'_c = \frac{1}{2}(1 - f^2)^2 = \xi_v'^4/2R'^4 \quad (7)$$

is the cost in condensation energy density,

$$\begin{aligned} F'_k &= [(i\kappa)^{-1}\nabla' + \mathbf{a}']f e^{-i\varphi}]^2 \\ &= (\xi_v'^2/\kappa R'^3)^2 + [K_1(R')/\kappa\xi_v'K_1(\xi_v')]^2 \end{aligned} \quad (8)$$

is the cost in kinetic energy density, and

$$F'_f = b'^2 = [K_0(R')/\kappa\xi_v'K_1(\xi_v')]^2 \quad (9)$$

is the cost in magnetic field energy density. Here, as in Ref. 1, primes denote dimensionless quantities, which correspond to measuring lengths in units of  $\lambda$ , the magnetic flux density in units of  $\sqrt{2}H_c = \kappa\varphi_0/2\pi\lambda^2$ , the vector potential in units of  $\sqrt{2}H_c\lambda = \kappa\varphi_0/2\pi\lambda$ , and the energy density in units of  $H_c^2/4\pi$ .

From Eqs. (6)–(9) we obtain the energy per unit length of vortex line

$$\varepsilon_1 = \int d^2\rho F = \varphi_0 H_{c1}/4\pi \quad (10)$$

or, expressed in dimensionless units,

$$H'_{c1} = H_{c1}/\sqrt{2} H_c = H'_{c1c} + H'_{c1k} + H'_{c1f} \quad (11)$$

where

$$H'_{c1c} = (\kappa/2) \int_0^\infty d\rho' \rho' F'_c = \kappa \xi_v'^2/8 \quad (12)$$

$$\begin{aligned} H'_{c1k} &= (\kappa/2) \int_0^\infty d\rho' \rho' F'_k \\ &= (8\kappa)^{-1} + [4\kappa K_1^2(\xi'_v)]^{-1} [K_0(\xi'_v)K_2(\xi'_v) - K_1^2(\xi'_v)] \end{aligned} \quad (13)$$

$$\begin{aligned} H'_{c1f} &= (\kappa/2) \int_0^\infty d\rho' \rho' F'_f \\ &= [4\kappa K_1^2(\xi'_v)]^{-1} [K_1^2(\xi'_v) - K_0^2(\xi'_v)] \end{aligned} \quad (14)$$

such that

$$H'_{c1} = \frac{1}{8}\kappa \xi_v'^2 + (8\kappa)^{-1} + K_0(\xi'_v)/2\kappa \xi'_v K_1(\xi'_v) \quad (15)$$

Helpful descriptions of the properties of modified Bessel functions are given in Refs. 5 and 6.

Since  $\xi'_v = \xi_v/\lambda$  is a variational parameter, we now minimize  $H'_{c1}$  with respect to  $\xi'_v$  and find that  $H'_{c1}$  is minimized when  $\xi'_v$  satisfies

$$\kappa = \sqrt{2} [1 - K_0^2(\xi'_v)/K_1^2(\xi'_v)]^{1/2}/\xi'_v \quad (16)$$

For large  $\kappa$ , we see that  $\xi'_v \approx \sqrt{2}/\kappa \ll 1$ , or since  $\kappa = \lambda/\xi$ ,  $\xi_v \approx \sqrt{2} \xi$ . Figure 2 is a plot of  $\xi_v/\xi = \kappa \xi'_v$  as a function of  $\kappa^{-1}$ . The value of this ratio ranges from 0.935 at  $\kappa = 1/\sqrt{2}$  to 1.414 at  $\kappa = \infty$ .

Shown as the solid curve in Fig. 3 is a corresponding plot of  $H'_{c1}$  vs.  $\kappa$ , where  $H'_{c1}$ , given by Eq. (15), is evaluated at the values of  $\xi'_v$  that satisfy Eq. (16). As expected, since  $H'_{c1}$  is obtained from a variational calculation, the resulting values of  $H'_{c1}$  are slightly greater than the corresponding values from numerical solutions of the Ginzburg–Landau equations,<sup>4</sup> shown as points. At  $\kappa = 1/\sqrt{2}$ , for example, our vortex model yields  $H'_{c1} = 0.732$ , about 4% higher than the exact Ginzburg–Landau result of 0.707.

Also shown in Fig. 3 as the dot-dash curve is a plot of  $b'_z(0) = b_z(0)/\sqrt{2} H_c$ , where  $b_z(0)$  is obtained from Eq. (4). At  $\kappa = 1/\sqrt{2}$ , our model yields  $b'_z(0) = 0.801$ , about 13% higher than the exact Ginzburg–Landau result of 0.707. The ratio  $b_z(0)/H_{c1}$  is shown as the dashed curve in Fig. 3. At  $\kappa = 1/\sqrt{2}$ , our model yields 1.095 for this ratio, which is to be compared with the exact Ginzburg–Landau result of unity. As can be shown from Eqs. (4), (15), and (16), the model also yields  $b_z(0)/H_{c1} \rightarrow 2$  in the limit  $\kappa \rightarrow \infty$ .

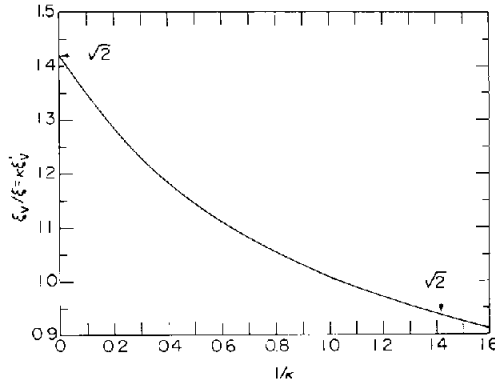


Fig. 2. The ratio  $\xi_v/\xi = \kappa\xi'_v = \kappa\xi_v^2/\lambda$  vs.  $\kappa^{-1}$ , calculated from Eq. (16), where  $\xi_v$  is the value of the variational core radius parameter that minimizes the Ginzburg-Landau free energy per unit length of the vortex,  $\xi$  is the Ginzburg-Landau coherence length,  $\lambda$  is the penetration depth, and  $\kappa = \lambda/\xi$ .

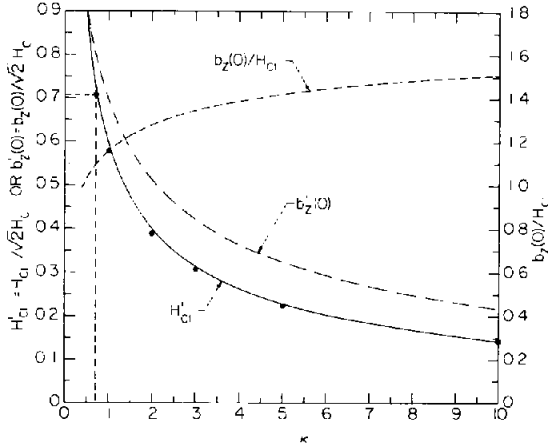


Fig. 3. Reduced lower critical field  $H'_{c1} = H_{c1}/\sqrt{2}H_c$  (solid curve), calculated from Eqs. (15) and (16), and reduced central magnetic flux density  $b'_2(0) = b_2(0)/\sqrt{2}H_c$  (dot-dash curve), calculated from Eqs. (4) and (16), vs.  $\kappa$  (left scale), and the resulting ratio  $b'_2(0)/H_{c1}$  vs.  $\kappa$  (right scale). The points show values of  $H'_{c1}$  given in Ref. 4 as exact numerical solutions of the Ginzburg-Landau equations.

The Fourier transform of the magnetic flux density for this model is

$$\begin{aligned}\tilde{b}_z(q) &= \int d^2\rho \, b_z(\rho) \exp(-i\mathbf{q} \cdot \boldsymbol{\rho}) \\ &= \varphi_0 K_1(Q\xi_v)/Q\lambda K_1(\xi_v/\lambda)\end{aligned}\quad (17)$$

where

$$Q = (q^2 + \lambda^{-2})^{1/2} \quad (18)$$

We note that, provided  $\xi_v/\lambda \ll 1$ ,

$$\tilde{b}_z(q) \approx \varphi_0/(1 + q^2\lambda^2) \quad (19)$$

when  $Q\xi_v \ll 1$ , and

$$\tilde{b}_z(q) \approx \varphi_0(\pi/2)^{1/2}(\xi_v/\lambda)^2(q\xi_v)^{-3/2} e^{-q\xi_v} \quad (20)$$

when  $Q\xi_v \gg 1$ . Figure 4 shows plots of the form factor  $\tilde{b}_z(q)/\varphi_0$  vs.  $q\lambda$  calculated from Eq. (17) for  $\kappa = 1, 2, 4, 8$ , and  $\infty$ . Since the form factors obtained from neutron diffraction studies of the vortex lattice in superconducting alloys<sup>7</sup> behave as in Fig. 4, we expect that Eq. (17) will prove useful in the analysis of such experiments.

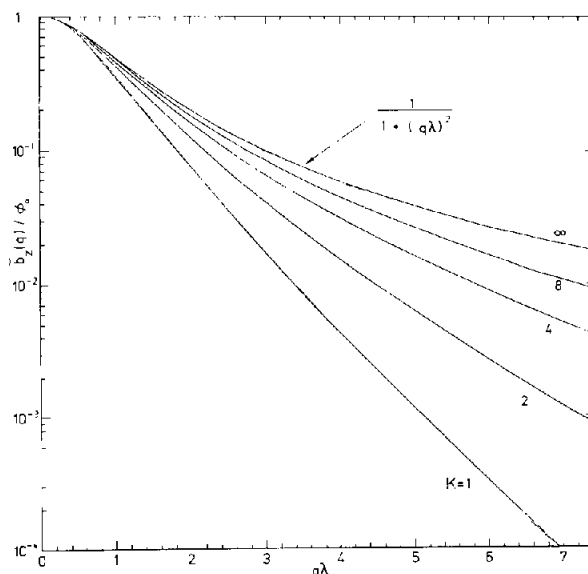


Fig. 4. Form factor or reduced Fourier transform of the magnetic flux density  $\tilde{b}_z(q)/\varphi_0$  vs.  $q\lambda$ , where  $q$  is the wave vector and  $\lambda$  the penetration depth, calculated from Eqs. (16) and (17) for  $\kappa = \lambda/\xi = 1, 2, 4, 8$ , and  $\infty$ .

The above model for a singly quantized vortex line in a type II superconductor is subject to the same theoretical limitations as the Ginzburg–Landau equations.<sup>3</sup> In particular, the temperature is restricted to the immediate vicinity of the superconducting transition temperature. Furthermore, our model might even be regarded as a corruption of the Ginzburg–Landau theory, since it is based upon a trial function that does not solve the Ginzburg–Landau equations exactly, but only approximately. Despite these limitations, we see the following virtues of the model:

1. The model is an improvement over the original London model in that it not only yields realistic behavior for the magnetic flux density and supercurrent density in the vortex core vicinity but also gives the London model behavior well outside the core.

2. The model is pedagogically useful in that it allows the contributions of the condensation energy, kinetic energy, and field energy to the energy per unit length of vortex line to be evaluated explicitly.

3. The model may well be helpful, even outside the claimed region of validity, in interpreting and parameterizing experimental data, such as neutron diffraction form factors.

The variational approach of this paper is analogous to that used by Amit and Gross<sup>8</sup> and by Fetter<sup>9</sup> to investigate the energy per unit length of a vortex in superfluid helium. In particular, our results in the limit  $\kappa \rightarrow \infty$  correspond to those of Fetter<sup>9</sup> for a vortex in helium, provided  $\lambda$  is replaced by  $R$ , the radius of the helium bucket, and  $\xi$  is replaced by  $a$ , the helium de Broglie wavelength or healing length.

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